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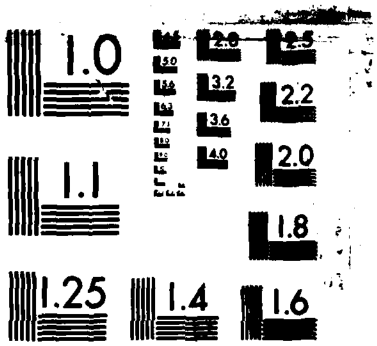

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STEADY-STATE PROBLEMS OF NONLINEAR ELECTRO-MAGNETO-THERMO-ELASTICITY

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ABSTRACT

The authors

In this paper ~~we study~~ studies the steady-state behavior of solids that can sustain mechanical, electromagnetic, and thermal effects. ~~We~~ examine a class of boundary-value problems for a quasilinear system of functional differential equations that is derived from a very general model for such materials. ~~We~~ They propose a physically reasonable constitutive theory which leaves this system amenable to modern methods of partial differential equations. The principal assumption is a modified version of the strong ellipticity condition. In Part I we prove existence results for the general system under some special physical assumptions (rigidity and hyperelasticity). ~~Our~~ The formulation admits non-local interactions caused by the magnetic "self-field" generated by the deformed, conducting body. In Part II we show the existence and regularity of solutions of a system of functional ordinary differential equations arising from a semi-inverse problem in a more comprehensive physical situation.

Keywords;

AMS (MOS) Subject Classifications: 73R05, 35J60

Key Words: electromagnetism, semi-inverse problem, compact operator equations, global existence, smooth solutions; polyconvex energy functions; electro-elastic coupling; magneto-elastic coupling; strong ellipticity ~~condition,~~ conducting rods; thermo-elastic coupling.

Work Unit Numbers 1 (Applied Analysis) and 2 (Physical Mathematics)

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*Some of the results reported here were developed in the doctoral dissertation of Rogers (1984).

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SIGNIFICANCE AND EXPLANATION

In this paper we study the steady-state behavior of solids that can sustain mechanical, electromagnetic, and thermal effects. Our goal is to formulate very general assumptions on the constitutive equations that are physically reasonable, yet leave the resulting mathematical problems tractable. The models we propose admit the following types of nonlinear behavior which are of particular interest when materials are subjected to large electromagnetic fields and sustain large currents.

1. Nonlinear Coupling: We investigate the abstract mathematical problems that occur when we assume a very general coupling of the various physical fields of the material. For instance, we assume that the dielectric displacement $\underset{\sim}{d}$ depends not only on the electric field $\underset{\sim}{e}$ but also on the deformation gradient $\underset{\sim}{F}$, temperature, temperature gradient $\underset{\sim}{g}$, and magnetic field $\underset{\sim}{h}$. We make similar assumptions about the stress tensor $\underset{\sim}{T}$, magnetic induction $\underset{\sim}{h}$, heat flux vector $\underset{\sim}{q}$, and electric current $\underset{\sim}{j}$. The assumption which makes such problems tractable is a modified version of the "strong ellipticity condition".

2. Nonlocal Self-Interactions: The Maxwell's equation

$$\text{curl } \underset{\sim}{h} = \underset{\sim}{j}$$

implies that a current in one part of a body will generate a magnetic field in a distant part of the body. Since we allow $\underset{\sim}{j}$ to depend on the entire list of independent variables, the magnetic field at any point will depend on the global values of the other variables. And, since all of our dependent variables depend on h, this problem will be spread to the entire system of equations. We use compactness method to handle these nonlocal problems.

In Part I we handle very general mathematical problems under special physical assumptions (rigidity and hyperelasticity) and in Part II we handle a special mathematical problem (a semi-inverse problem) under general physical conditions.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

Contents

STEADY-STATE PROBLEMS OF NONLINEAR ELECTRO-MAGNETO-THERMO-ELASTICITY*

Robert C. Rogers[1] and Stuart S. Antman[2]

Part I. The General Theory

1. Introduction

In this paper we study a class of boundary value problems for a quasilinear system of functional-differential equations describing the steady-state behavior of solids that can sustain mechanical, electromagnetic, and thermal effects. We treat partial differential equations in Part I and ordinary differential equations in Part II. Our primary goals are to show that there is a simple way to formulate the governing equations, which illuminates the physics and promotes the analysis of the equations, to actually analyze important classes of problems, and to contribute to the development of an effective constitutive theory for such materials by showing how our physically and mathematically natural constitutive restrictions support existence and regularity theories for our problems. The problems we study are simple enough to be tractable, interesting enough to possess a very rich class of solutions, and yet complicated enough to require new approaches, both in the formulation and treatment of electromagnetism in solids and in the use of techniques of nonlinear analysis.

Our constitutive equations give the stress, heat flux, dielectric displacement, magnetic induction, and electric current as arbitrary functions of

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the deformation gradient, temperature, temperature gradient, electric field, and magnetic field. These constitutive functions must of course be invariant under rigid motions. In order to reduce the governing equation to ordinary differential equations for our semi-inverse problems of Part II, we further require that the bodies under study have some material symmetry. Our basic constitutive assumption is that the constitutive equations satisfy the Strong Ellipticity Condition. Originating in the theory of partial differential equations, this condition proves to be eminently natural on physical grounds. Indeed, this condition, when precisely formulated, is equivalent to the requirement that each component of the dependent constitutive variables is a strictly increasing function of the corresponding component of the independent constitutive variables (when the other components of the independent variables are held fixed). Roughly speaking, a typical consequence of this assumption is that a change in the temperature gradient produces a far more pronounced change in the heat flux vector than it does in the stress, dielectric displacement, and magnetic induction. Thus the Strong Ellipticity Condition implies a very mild uncoupling in the constitutive equations. True uncouplings, such as the independence of stress, dielectric displacement, and magnetic induction on the temperature gradient, may be interpreted as consequences of the Clausius-Duhem version of the Second Law of Thermodynamics. But we have no need for such true uncouplings anywhere in our analysis. Indeed, wherever the Clausius-Duhem inequality is more restrictive than the Strong Ellipticity Condition, we have no need for its consequences, and wherever it is less restrictive, it is inadequate for our needs. We supplement the Strong Ellipticity Condition with compatible growth conditions.

Rather than adhering to the classical tradition, (followed at least in part by Toupin (1956), Fano, Chu, & Adler (1960), Penfield & Haus (1967), and DeGroot

& Suttorp (1972) and others), of deriving or motivating the constitutive equations of electromagnetism from discrete microscopic models, we employ the phenomenological approach of continuum mechanics and simply lay down general constitutive laws. We thereby gain great economy and generality in our formulation of electromagnetism. (Cf. Truesdell & Toupin (1960).)

For simplicity and clarity in our mathematical analysis, it is crucial not only that we give Maxwell's equations a material (Lagrangian) formulation, but also that we introduce new variables in place of the dielectric displacement, magnetic induction, etc. In this regard we are merely extending to the theory of electromagnetism in deformable media the methodology that has proved most natural and successful for boundary value problems of nonlinear elasticity.

In the overwhelming majority of texts on continuum mechanics the emphasis placed on the spatial formulation of the governing equations and, in particular, on the use of the Cauchy stress tensor overshadows that placed on the material formulation and on the use of the Piola-Kirchhoff stress tensors. (We define these stress tensors in Section 3.) The reason for this emphasis is largely historical: The two most highly developed branches of continuum mechanics are Newtonian fluids and linear elasticity. The constitutive equations for a homogeneous Newtonian fluid are especially simple in a spatial formulation. Moreover, in this formulation the constraint of incompressibility (valid for liquids) has an elegant characterization as the linear equation expressing the vanishing of the divergence of the velocity field defined over points in space. (In contrast, incompressibility is characterized in a material formulation by the nonlinear equation requiring the Jacobian of the deformation gradient to equal unity.) For problems involving nonhomogeneous fluids or fluids with free surfaces, there are compensating disadvantages requiring some version of a material formulation, possibly disguised, for their successful treatment. In linear elasticity there is no distinction between material and spatial formulations, although the derivation of these equations from a nonlinear spatial formulation is much more difficult to carry out than that from a nonlinear material formulation. The advantages of a material formulation are evident for the boundary value problems of nonlinear solid mechanics: (i) The prescription of constitutive equation for the first Piola-Kirchhoff stress tensor in terms of the past history of deformation is natural and does not suffer from complications due to nonhomogeneity. (ii) The governing equations are posed on a <u>fixed</u> region of space, the region occupied by the body in a reference configuration, rather than on the unknown and possibly moving region actually occupied by the body. These factors have not, however, proved to be compelling in shifting the emphasis of texts towards material formulations because there have been so few studies of nonlinear boundary value problems of solid mechanics. (Cf. Antman (1978, 1979, 1983), Ball (1977, 1982).)

Maxwell's equations have been posed almost exclusively in spatial coordinates because the most important case of the vacuum can be posed in no other way and because in the second most important case of a rigid medium there is no essential distinction between material and spatial coordinates. Moreover, the most actively cultivated field of electromagnetism in deformable media is that of magnetohydrodynamics. For the reasons mentioned in our comments on fluid dynamics, many problems for this theory are most easily set in spatial coordinates. Only recently has the use of material coordinates begun to appear in treatments of electromagnetism in media. (Cf. Walker, Pipkin, & Rivlin (1965), Hutter (1975), Pao & Hutter (1975), Hutter & van de Ven (1978), McCarthy & Tiersten (1978), and Maugin (1981).) These authors have also introduced new fields suitable for material coordinates in place of the classical fields.

The large deformation of solids in the presence of large electromagnetic fields is a problem of growing technological importance (cf. Moon (1978, 1984)). Awareness of this importance is evidenced by the number of papers recently devoted to this subject. (Cf. Parkus (1979), Ambartsumian (1982), Maugin (1983).) To our knowledge, ours is the first mathematical analysis of general nonlinear boundary value problems in this area.

Much of the previous work in the electromagnetism of deformable solids can be divided into two general areas: the development of a general theory governing such media and the solution of specific nonlinear problems. General theories of the dynamics of deformable solids have been proposed by Fano, Chu & Adler (1960), Toupin (1963), Dixon & Eringen (1965), Pao & Hutter (1975), and Maugin & Eringen (1977). These developments consist in the derivation of some form of Maxwell's equation and associated forms of the electric body force, body couple, and internal energy supply from some discrete model of the material. Comparisons of various theories are to be found in Penfield & Haus (1967), DeGroot & Suttorp (1972), Hutter & van de Ven (1978), and Pao (1978). Our work is more closely related to the static theories of Toupin (1956) and Brown (1966). Specific problems for general nonlinear dielectrics were solved by Toupin (1956), Eringen (1963), Verma (1964), Pipkin & Rivlin (1960), and Singh & Pipkin (1966) via the inverse methods of modern nonlinear elasticity. (Singh & Pipkin also provide a review of the earlier work.) There is an extensive literature on specific nonlinear materials with polynomial constitutive equations and associated problems (cf. Jordan & Eringen (1964) and Pipkin & Rivlin (1966)). Maugin (1981) reviews the modern work on wave motion in magnetizable deformable solids and includes both general nonlinear and specific (approximate) constitutive equations.

2. Notation

The Euclidean 3-space $\mathbb{E}^3$ is defined to be abstract 3-dimensional real inner-product space. Its inner product, the dot product, is the natural source of the geometric properties of the space. We interpret $\mathbb{E}^3$ as physical space. We distinguish $\mathbb{E}^3$ from $\mathbb{R}^3$, the space of triples of real numbers equipped with any norm (which is necessarily topologically equivalent to the Euclidean norm). But we assign no natural geometrical significance to the norm on $\mathbb{R}^3$.

Vectors, which we define to be elements of $\mathbb{E}^3$, and vector-valued functions are denoted by bold-face, lower-case Latin letters. Second-order tensors, which form the space Lin of linear operators from $\mathbb{E}^3$ into itself, are denoted by bold-face upper-case Latin letters. The subspace of Lin consisting of symmetric second-order tensors is denoted Sym. Its subset of positive-definite tensors is denoted Psym. The group SL(3) of all members of Lin with positive determinant is denoted Lin$^+$. Scalars and scalar-valued functions are denoted with light-faced letters. Elements of $\mathbb{R}^n$ for $n > 1$ and functions with values in $\mathbb{R}^n$ are denoted by bold-face sanserif lower-case Latin letters and by bold-face lower-case Greek letters.

We employ the dyadic notation of Gibbs (cf. Gibbs & Wilson (1901)), which we now describe. (This notation is both admirably suited for treatment of problems in curvilinear coordinates and completely compatible with modern invariant formulations of linear algebra in $\mathbb{E}^3$.) The dot product of vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is denoted $\underset{\sim}{u}\cdot\underset{\sim}{v}$. The cross product of two vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is denoted $\underset{\sim}{u}\wedge\underset{\sim}{v}$. The value of the second order tensor $\underset{\sim}{A}$ at $\underset{\sim}{u}$ is denoted $\underset{\sim}{A}\cdot\underset{\sim}{u}$. The transpose $\underset{\sim}{A}^*$ of $\underset{\sim}{A}$ is defined by $\underset{\sim}{v}\cdot(\underset{\sim}{A}\cdot\underset{\sim}{u}) = \underset{\sim}{u}\cdot(\underset{\sim}{A}^*\cdot\underset{\sim}{v})$ for all $\underset{\sim}{u}$ and $\underset{\sim}{v}$. We accordingly write $\underset{\sim}{A}^*\cdot\underset{\sim}{v} = \underset{\sim}{v}\cdot\underset{\sim}{A}$. A is symmetric if $\underset{\sim}{A} = \underset{\sim}{A}^*$ and skew if $\underset{\sim}{A} = -\underset{\sim}{A}^*$. If $\underset{\sim}{A}$ is skew, there is a unique vector $\underset{\sim}{a}$, called the axial vector of $\underset{\sim}{A}$, such that $\underset{\sim}{A}\cdot\underset{\sim}{v} = \underset{\sim}{a}\wedge\underset{\sim}{v}$ for all $v \in \mathbb{E}^3$. The dyadic product $\underset{\sim}{u}\underset{\sim}{v}$ of

vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is defined to be the second-order tensor satisfying $(\underset{\sim}{u}\underset{\sim}{v})\cdot\underset{\sim}{w} = (\underset{\sim}{v}\cdot\underset{\sim}{w})\underset{\sim}{u}$ for all $\underset{\sim}{w}$. Thus $(\underset{\sim}{u}\underset{\sim}{v})^* = \underset{\sim}{v}\underset{\sim}{u}$ and $\mathrm{tr}(\underset{\sim}{v}\underset{\sim}{u}) = \underset{\sim}{u}\cdot\underset{\sim}{v}$ where tr denotes trace. The <u>product</u> $(\underset{\sim}{A}\cdot\underset{\sim}{B})$ of tensors is defined by $(\underset{\sim}{A}\cdot\underset{\sim}{B})\cdot\underset{\sim}{v} = \underset{\sim}{A}\cdot(\underset{\sim}{B}\cdot\underset{\sim}{v})$ for all $\underset{\sim}{v}$. Thus $\underset{\sim}{A}\cdot(\underset{\sim}{u}\underset{\sim}{v}) = (\underset{\sim}{A}\cdot\underset{\sim}{u})\underset{\sim}{v}$ and $(\underset{\sim}{u}\underset{\sim}{v})\cdot(\underset{\sim}{A}) = \underset{\sim}{u}(\underset{\sim}{v}\cdot\underset{\sim}{A})$. We set $\underset{\sim}{A}:\underset{\sim}{B} = \mathrm{tr}(\underset{\sim}{A}\cdot\underset{\sim}{B}^*)$. Hence $\mathrm{tr}\underset{\sim}{A} = \underset{\sim}{I}:\underset{\sim}{A} = \underset{\sim}{A}:\underset{\sim}{I} = \underset{\sim}{A}^*:\underset{\sim}{I} = \mathrm{tr}\underset{\sim}{A}^*$, $(\underset{\sim}{u}\underset{\sim}{v}):(\underset{\sim}{x}\underset{\sim}{y}) = (\underset{\sim}{u}\cdot\underset{\sim}{x})(\underset{\sim}{v}\cdot\underset{\sim}{y})$, and $\underset{\sim}{A}:(\underset{\sim}{u}\underset{\sim}{v}) = \underset{\sim}{u}\cdot\underset{\sim}{A}\cdot\underset{\sim}{v} = (\underset{\sim}{u}\underset{\sim}{v}):\underset{\sim}{A}$. ($\underset{\sim}{I}$ denotes the identity tensor.) It is easy to see that ":" is an inner product on Lin. We accordingly define the <u>norm</u> $|\underset{\sim}{A}|$ of $\underset{\sim}{A}$ by $|\underset{\sim}{A}| = \sqrt{\underset{\sim}{A}:\underset{\sim}{A}}$. If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are unit vectors, then $|\underset{\sim}{a}\underset{\sim}{b}| = 1$ so that $\underset{\sim}{a}\underset{\sim}{b}$ is a unit tensor. In this case we can represent any tensor $\underset{\sim}{A}$ by the orthogonal decomposition $\underset{\sim}{A} = (\underset{\sim}{a}\cdot\underset{\sim}{A}\cdot\underset{\sim}{b})\underset{\sim}{a}\underset{\sim}{b} + [\underset{\sim}{A} - (\underset{\sim}{a}\cdot\underset{\sim}{A}\cdot\underset{\sim}{b})\underset{\sim}{a}\underset{\sim}{b}]$ where $\underset{\sim}{a}\cdot\underset{\sim}{A}\cdot\underset{\sim}{b}$ is the <u>component of</u> $\underset{\sim}{A}$ <u>along</u> $\underset{\sim}{a}\underset{\sim}{b}$ and $[\underset{\sim}{A} - (\underset{\sim}{a}\cdot\underset{\sim}{A}\cdot\underset{\sim}{b})\underset{\sim}{a}\underset{\sim}{b}]$ is the <u>projection of</u> $\underset{\sim}{A}$ <u>onto the orthogonal complement of</u> $\underset{\sim}{a}\underset{\sim}{b}$. If $\{\underset{\sim}{a}_k\}$ and $\{\underset{\sim}{b}_j\}$ are each bases for $\mathbb{E}^3$, then $\{\underset{\sim}{a}_k\underset{\sim}{b}_j\}$ is a basis for Lin. Thus we can use all of our dyadic identities to construct the familiar componential formulas for all the expressions we have introduced in the invariant form. Repeated indices are summed over their obvious ranges.

The <u>(Gâteaux) differentials</u> of $\underset{\sim}{u} \mapsto \underset{\sim}{f}(\underset{\sim}{u})$ at $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ and of $\underset{\sim}{U} \mapsto \underset{\sim}{F}(\underset{\sim}{U})$ at $\underset{\sim}{A}$ in the direction of $\underset{\sim}{B}$ are defined to be the vector $[\partial\underset{\sim}{f}(\underset{\sim}{a})/\partial\underset{\sim}{u}]\cdot\underset{\sim}{b}$ and the tensor $[\partial\underset{\sim}{F}(\underset{\sim}{A})/\partial\underset{\sim}{U}]\cdot\underset{\sim}{B}$ given by

(2.1) $$[\partial f(\underset{\sim}{a})/\partial\underset{\sim}{u}]\cdot\underset{\sim}{b} = \frac{d}{dt}\,\underset{\sim}{f}(\underset{\sim}{a} + t\underset{\sim}{b})\Big|_{t=0} ,$$

(2.2) $$[\partial\underset{\sim}{F}(\underset{\sim}{A})/\partial\underset{\sim}{U}]\cdot\underset{\sim}{B} = \frac{d}{dt}\,\underset{\sim}{F}(\underset{\sim}{A} + t\underset{\sim}{B})\Big|_{t=0} .$$

Other differentials are defined similarly. If $\underset{\sim}{U} \mapsto \underset{\sim}{F}(\underset{\sim}{U})$ and $\underset{\sim}{V} \mapsto \underset{\sim}{G}(\underset{\sim}{V})$ are (Fréchet) differentiable, then $\underset{\sim}{V} \mapsto \underset{\sim}{F}(\underset{\sim}{G}(\underset{\sim}{V})) = \underset{\sim}{H}(\underset{\sim}{V})$ is also, and its differential satisfies the <u>chain rule</u>:

(2.3) $$[\partial\underset{\sim}{H}(\underset{\sim}{A})/\partial\underset{\sim}{V}]:\underset{\sim}{B} = [\partial\underset{\sim}{F}(\underset{\sim}{G}(\underset{\sim}{A}))/\partial\underset{\sim}{U}]:\{[\partial\underset{\sim}{G}(\underset{\sim}{A})/\partial\underset{\sim}{V}]:\underset{\sim}{B}\} .$$

(The braces can be omitted from the right side of (2.3).) Our notational scheme embodied in (2.1) and (2.2) causes the chain rule (2.3) to have a form analogous

to that for scalar functions. As we shall see in the next chapter this virtue is counterbalanced by the increased complexity of defining and representing the action of the classical differential operators grad, div, curl on tensor functions.

We denote n copies of a function space X by X itself. The distinction will be clear from the context: Thus a statement of the form $\underset{\sim}{w} \in L_p(B)$ is to imply that this $L_p(B)$ is the space of all measurable vector-valued functions

$$\mathbb{E}^3 \supset B \ni \underset{\sim}{z} \mapsto \underset{\sim}{w}(\underset{\sim}{z}) \in \mathbb{E}^3 \quad \text{such that} \quad \int_B [\underset{\sim}{w}(z)\cdot\underset{\sim}{w}(\underset{\sim}{z})]^{p/2} dv(\underset{\sim}{z}) < \infty .$$

The norm of a Banach space X is denoted $\|\cdot, X\|$.

3. Formulation of the Governing Equations

In this and the next section we formulate the equations for steady-state problems of electro-magneto-thermo-elasticity. There are several different theories that are at least formally equivalent in the classical nonrelativistic setting we employ. (Cf. Hutter & van de Ven (1978) and Pao (1978).) Comparisons of the various theories is made difficult by the fact that the same symbol used in different theories has different meanings. Fortunately, the mathematical form of the governing equations expressing the balance of linear momentum, the balance of energy, Maxwell's laws, and the conservation of charge is the same for all these theories. We shall refer to the various fields that occur in our equations by their traditional names, realizing that their precise physical significance inheres in the slots they occupy in the equations for a specific theory.

To make our presentation as transparent as possible, we assume that all the functions and boundaries that appear are smooth enough for all the classical operations that appear to make sense. (A careful treatment of these issues without such blanket smoothness assumptions can be modelled on that of Antman & Osborn (1979).) Of course, we abandon this optimistic formalism when we afterwards analyze our specific boundary value problems.

We identify a <u>material body</u> with the closure $\mathcal{B}$ of a domain in $\mathbb{E}^3$ and we identify <u>material points</u> of the body with their positions $\underset{\sim}{z}$ in $\mathcal{B}$. For each $\underset{\sim}{z}$ in $\mathcal{B}$ let $\underset{\sim}{\chi}(\underset{\sim}{z})$ denote the position of $\underset{\sim}{z}$ in some deformed configuration. The <u>(transposed) deformation gradient</u> $\underset{\sim}{F}$ and the <u>right Cauchy-Green deformation tensor</u> $\underset{\sim}{C}$ for the configuration $\underset{\sim}{\chi}$ are defined by

$$(3.1) \qquad \underset{\sim}{F} = \partial\underset{\sim}{\chi}/\partial\underset{\sim}{z}, \quad \underset{\sim}{C} = \underset{\sim}{F}^{*}\cdot\underset{\sim}{F} \cdot$$

We require that no two distinct material points simultaneously occupy the same position in a given configuration. Thus each map $\underset{\sim}{\chi}$ must be one-to-one. Since this global condition is so difficult to treat, we ignore it and content

ourselves with the local condition that the deformation $\underset{\sim}{\chi}$ merely preserve orientation, i.e., that

$$\det \underset{\sim}{F} > 0 , \tag{3.2}$$

where det denotes the determinant.

Let $\tilde{\lambda}(\underset{\sim}{y})$ denote the <u>logarithm of the absolute temperature</u> at position $\underset{\sim}{y}$ in space. (It is finite-valued if and only if the absolute temperature is positive-valued.) We set $\tilde{\underset{\sim}{g}}(\underset{\sim}{y}) = \partial\tilde{\lambda}(\underset{\sim}{y})/\partial\underset{\sim}{y}$. Let $\tilde{\underset{\sim}{e}}(\underset{\sim}{y})$ and $\tilde{\underset{\sim}{h}}(\underset{\sim}{y})$ denote the <u>electric</u> and <u>magnetic fields</u> at $\underset{\sim}{y}$. We set

$$\lambda(\underset{\sim}{z}) \equiv \tilde{\lambda}(\underset{\sim}{\chi}(\underset{\sim}{z})),\ \underset{\sim}{g}(\underset{\sim}{z}) \equiv \partial\lambda(\underset{\sim}{z})/\partial\underset{\sim}{z} \equiv \tilde{\underset{\sim}{g}}(\underset{\sim}{\chi}(\underset{\sim}{z}))\cdot\underset{\sim}{F}(\underset{\sim}{z}) ,$$
$$\underset{\sim}{e}(\underset{\sim}{z}) = \tilde{\underset{\sim}{e}}(\underset{\sim}{\chi}(\underset{\sim}{z}))\cdot\underset{\sim}{F}(\underset{\sim}{z}),\ \underset{\sim}{h}(\underset{\sim}{z}) = \tilde{\underset{\sim}{h}}(\underset{\sim}{\chi}(\underset{\sim}{z}))\cdot\underset{\sim}{F}(\underset{\sim}{z}) . \tag{3.3}$$

$\underset{\sim}{g}$, $\underset{\sim}{e}$, and $\underset{\sim}{h}$ are the <u>material logarithmic temperature gradient, electric field</u>, and <u>magnetic field</u>. (We shall soon see that $\underset{\sim}{e}$ and $\underset{\sim}{h}$ can be represented in terms of gradients. Consequently they transform in (3.3) just like $\underset{\sim}{g}$.)

Let $\tilde{\underset{\sim}{T}}(\underset{\sim}{y})$ denote the <u>effective Cauchy stress</u>, i.e., the sum of the mechanical Cauchy stress and the Maxwell stress. $\tilde{\underset{\sim}{T}}(\underset{\sim}{y})$ measures force per unit actual area at $\underset{\sim}{y}$. (There are several versions of $\tilde{\underset{\sim}{T}}$, depending on alternative representations and decompositions of the Lorentz force and torque.) Let $\tilde{\underset{\sim}{q}}(\underset{\sim}{y})$ denote the <u>heat flux</u> per unit actual area at $\underset{\sim}{y}$. Let $\tilde{\underset{\sim}{d}}(\underset{\sim}{y})$, $\tilde{\underset{\sim}{b}}(\underset{\sim}{y})$, $\tilde{\underset{\sim}{j}}(\underset{\sim}{y})$ be the <u>dielectric displacement, magnetic induction</u>, and <u>current density</u> at $\underset{\sim}{y}$. Then we introduce material versions of these fields by

$$\underset{\sim}{T}(\underset{\sim}{z})^{*} = \det \underset{\sim}{F}(z)\underset{\sim}{F}^{-1}(\underset{\sim}{z})\cdot\tilde{\underset{\sim}{T}}(\underset{\sim}{\chi}(\underset{\sim}{z}))^{*} ,$$
$$\underset{\sim}{q}(\underset{\sim}{z}) = \det \underset{\sim}{F}(\underset{\sim}{z})\underset{\sim}{F}^{-1}(\underset{\sim}{z})\cdot\tilde{\underset{\sim}{q}}(\underset{\sim}{\chi}(z)),\quad \text{etc.} \tag{3.4}$$

$\underset{\sim}{T}$ is the <u>effective first Piola-Kirchhoff stress</u>.

For simplicity let us assume that the body force and heat source have purely electromagnetic origin. Then the local form of the balance of forces, the balance of energy, and Maxwell's equations for a steady state are

$$\text{Div}\,\mathbf{T} + \mathbf{f} = \mathbf{0}\,, \tag{3.5}$$

$$\text{Div}\,\mathbf{q} + \mathbf{j}\cdot\mathbf{e} = 0\,, \tag{3.6}$$

$$\text{Div}\,\mathbf{d} = \sigma\,, \tag{3.7}$$

$$\text{Div}\,\mathbf{b} = 0\,, \tag{3.8}$$

$$\text{Curl}\,\mathbf{e} = \mathbf{0}\,, \tag{3.9}$$

$$\text{Curl}\,\mathbf{h} = \mathbf{j}\,. \tag{3.10}$$

The material divergence Div of a tensor is defined by Green's Theorem

$$\int_{\partial P} \mathbf{T}\cdot\mathbf{n}\,da = \int_{P} \text{Div}\,\mathbf{T}\,dv \tag{3.11}$$

where $P \subset \mathcal{B}$ and $\mathbf{n}$ is the unit outer normal to P. In (3.5) $\mathbf{f}$ represents body forces of electromagnetic origin not absorbed by the Maxwell stress. Since every term in the usual prescriptions of the Lorentz force is a divergence, we could absorb this force entirely into the Maxwell stress and hence into the effective stress. We accordingly take $\mathbf{f} = \mathbf{0}$. (Cf. Hutter & van de Ven (1978).) The term $\mathbf{j}\cdot\mathbf{e}$ in (3.6) is the <u>Joule heating</u>. In (3.7) σ represents the <u>free charge</u>. We regard it as an assigned function of $\mathbf{z}$.

The balance of torque has the local form

$$\mathbf{L} = \mathbf{T}\cdot\mathbf{F}^{*} - \mathbf{F}\cdot\mathbf{T}^{*} \tag{3.12}$$

where $\mathbf{L}$ is a skew tensor depending upon the electromagnetic fields and the choice of the Maxwell stress tensor. We assume that (3.12) is identically satisfied when the constitutive functions, to be introduced in the next section, are substituted into (3.12). Hutter & van de Ven (1978) show that it is permissible to take $\mathbf{L} = \mathbf{0}$.

Equations (3.9) and (3.10) imply that there exist scalar functions φ and ψ, called the <u>electric</u> and <u>magnetic scalar potentials</u> such that

$$\mathbf{e} = \partial\varphi/\partial\mathbf{z}\,, \tag{3.13}$$

(3.14) $$\underset{\sim}{h}(\underset{\sim}{z}) = \partial\psi(\underset{\sim}{z})/\partial\underset{\sim}{z} + \int_B [\underset{\sim}{j}(\underset{\sim}{u}) \wedge (\underset{\sim}{u} - \underset{\sim}{z})]|\underset{\sim}{u} - \underset{\sim}{z}|^{-3}dv(\underset{\sim}{u}) ,$$

as is shown in standard books on electromagnetism. (These formulas justify the remarks following (3.3).)

4. Constitutive Equations

Of all the variables that have appeared only σ is prescribed. The remaining variables are related by constitutive equations. As our independent constitutive variables we choose

$$\underline{\Gamma} \equiv (\underset{\sim}{F}, \underset{\sim}{g}, \lambda, \underset{\sim}{e}, \underset{\sim}{h}) \tag{4.1}$$

because they are physically reasonable and mathematically convenient. The domain of $\underset{\sim}{F}$ is Lin^+, the domain of $\underset{\sim}{g}, \underset{\sim}{e}, \underset{\sim}{h}$ is $\mathbb{E}^3$, and the domain of λ is $\mathbb{R}$. We first suppose that $\underset{\sim}{q}, \underset{\sim}{d}, \underset{\sim}{b}, \underset{\sim}{j}$ depend on these variables and on $\underset{\sim}{z}$. Thus $\underset{\sim}{j} \cdot \underset{\sim}{e}$, appearing in (3.6), likewise depends on (4.1). We finally prescribe $\underset{\sim}{T}$ and $\underset{\sim}{L}$ to depend on (4.1) and $\underset{\sim}{z}$ so that they satisfy (3.12). Henceforth we shall have no need for (3.12). Thus we have constitutive functions $\hat{\underset{\sim}{T}}, \hat{\underset{\sim}{q}}, \hat{\underset{\sim}{d}}, \hat{\underset{\sim}{b}}, \hat{\underset{\sim}{j}}, \hat{\underset{\sim}{p}}$ such that

$$\underset{\sim}{T}(\underset{\sim}{z}) = \hat{\underset{\sim}{T}}(\underline{\Gamma}(\underset{\sim}{z}), \underset{\sim}{z}), \quad \text{etc.} \tag{4.2}$$

The functions $\hat{\underset{\sim}{T}}$, etc., must be invariant under rigid motions, i.e., be frame-indifferent. (Cf. Truesdell & Noll (1965).) We do not pause to exhibit the specific representations of the constitutive functions that are necessary and sufficient for frame-indifference because we have no need for them in our analysis.

For simplicity, we assume that our constitutive functions are continuously differentiable.

5. Potentials

It is convenient in our analysis to employ the potentials φ and ψ instead of $\underset{\sim}{e}$ and $\underset{\sim}{h}$ as the fundamental variables defining the electromagnetic state. $\underset{\sim}{e}$ is expressed as the gradient of φ in (5.15). If the current $\underset{\sim}{j} = \underset{\sim}{0}$, then $\underset{\sim}{h}$ is likewise expressed as the gradient of ψ by (5.16). We seek conditions ensuring that $\underset{\sim}{h}$ can be expressed in terms of

(5.1) $$(\underset{\sim}{F}, g, \lambda, \partial\varphi/\partial\underset{\sim}{z}, \partial\psi/\partial\underset{\sim}{z}) \equiv \underline{\Delta}$$

when the current is not zero. Note that each entry in $\underline{\Delta}$ except λ is a gradient. Let us substitute our constitutive equation for $\underset{\sim}{j}$ into (3.14) to obtain

(5.2) $$\underset{\sim}{h}(\underset{\sim}{z}) - \partial\psi(\underset{\sim}{z})/\partial\underset{\sim}{z} = \int_{\mathcal{B}} [\hat{\underset{\sim}{j}}(\underset{\sim}{h}(\underset{\sim}{x}), \underline{\Sigma}(\underset{\sim}{x}), \underset{\sim}{x}) \wedge (\underset{\sim}{x} - \underset{\sim}{z})] |\underset{\sim}{x} - \underset{\sim}{z}|^{-3} dv(\underset{\sim}{x}) \equiv \underset{\sim}{k}(\underset{\sim}{h}, \underline{\Sigma})(\underset{\sim}{z})$$

where $\underline{\Sigma} \equiv (\underset{\sim}{F}, g, \lambda, \underset{\sim}{e})$. Now in the classical form of Ohm's Law, $\hat{\underset{\sim}{j}}$ depends only on the electric field. More generally, if $\hat{\underset{\sim}{j}}$ is independent of $\underset{\sim}{h}$, then (5.2) gives an explicit representation for $\underset{\sim}{h}$ in terms of $\underline{\Delta}$. There are a variety of results available for the case that $\hat{\underset{\sim}{j}}$ depends on $\underset{\sim}{h}$. Typical is the following:

<u>5.3. Theorem.</u> <u>Let</u> $\alpha > 1$ <u>and let</u> $\mathcal{B}$ <u>lie in the ball</u> B_γ <u>of radius</u> γ and center $\underset{\sim}{0}$. <u>Let</u> $\underline{\Delta}$ <u>be fixed in</u> $L_\alpha(\mathcal{B})$. <u>Suppose that there are positive numbers</u> μ, θ, ζ with $3\zeta < \alpha$ <u>such that</u>

(5.4) $$|\hat{\underset{\sim}{j}}(\underline{\Gamma}, \underset{\sim}{z})| \le \mu(1 + |\underline{\Gamma}|^{1+\zeta}) ,$$

(5.5) $$|\partial\hat{\underset{\sim}{j}}(\underline{\Gamma}, \underset{\sim}{z})/\partial\underset{\sim}{h}| \le \theta(1 + |\underline{\Gamma}|^{\zeta}) .$$

<u>If</u> γ <u>and</u> θ <u>are small enough, then</u> (5.2) <u>has a unique solution of the form</u>

(5.6) $$\underset{\sim}{h}(\underset{\sim}{z}) = \partial\psi(\underset{\sim}{z})/\partial\underset{\sim}{z} + \hat{\underset{\sim}{k}}(\underline{\Delta})(\underset{\sim}{z})$$

<u>where</u> $L_\alpha(\mathcal{B}) \ni \underline{\Delta} \mapsto \hat{\underset{\sim}{k}}(\underline{\Delta})(\cdot) \in L_\alpha(\mathcal{B})$ <u>is continuous</u> and compact.

<u>Proof</u>. It suffices to take $\mathcal{B} = B_\gamma$. We use the following amalgamation of results of Sobolev and Kantorovich (cf. Sobolev 1950, §6) and Kantorovich & Akilov (1977, Chap. XI, §3): <u>If</u> $f \in L_\beta(B_\gamma)$, <u>with</u> $\beta > 1$, <u>then there is a</u>

continuous function $\gamma \to \kappa(\beta,\gamma)$ that strictly increases from 0 to ∞ as γ increases from 0 to ∞ such that

(5.7a) $$\|Kf, L_\nu(B_\gamma)\| \leq \kappa(\beta,\gamma)\|f, L_\beta(B_\gamma)\|$$

where

(5.7b) $$(Kf)(z) \equiv \int_{B_\gamma} \frac{f(\underset{\sim}{x})}{|\underset{\sim}{x} - \underset{\sim}{z}|^2} dv(\underset{\sim}{z}) ,$$

$\nu = \infty$ if $\beta > 3$, $\nu < 3\beta/(3 - \beta)$ if $\beta \leq 3$. Moreover K is compact (and continuous) from $L_\beta(B_\gamma)$ to $L_\nu(B_\gamma)$.

We wish to show that $\underset{\sim}{h} \mapsto \underset{\sim}{k}(\underset{\sim}{h},\underset{\sim}{\Sigma},\cdot)$ is a contraction from $L_\alpha(B_\gamma)$ to itself. Let $\underset{\sim}{h} \in L_\alpha(B_\gamma)$. We first identify f of (5.7) with (the components of) $\hat{\underset{\sim}{j}}(\underset{\sim}{h},\underline{\Sigma})(\cdot)$ and chose $\beta = \alpha(1 + \zeta)^{-1}$. (Then $\beta \leq 3$ if and only if $\alpha - 3\zeta \leq 3$.) Then (5.4) ensures that $\hat{\underset{\sim}{j}} \in L_\alpha(B_\gamma)$. Since $3\beta(3 - \beta)^{-1} > \alpha$ when $\beta \leq 3$, we can take $\nu = \alpha$ in this case. Thus (5.7) implies that $\underset{\sim}{h} \mapsto \underset{\sim}{k}(\underset{\sim}{h},\underline{\Sigma})(\cdot)$ maps $L_\alpha(B_\gamma)$ into itself.

We now show that $\underset{\sim}{h} \mapsto \underset{\sim}{k}(\underset{\sim}{h},\underline{\Sigma},\cdot)$ is a contraction. Let $\underset{\sim}{h}_1,\underset{\sim}{h}_2 \in L_\alpha(B_\gamma)$, $\delta\underset{\sim}{h} \equiv \underset{\sim}{h}_1 - \underset{\sim}{h}_2$. Then

(5.8) $$|\delta\underset{\sim}{k}(\underset{\sim}{h}_1,\underset{\sim}{h}_2,\underline{\Sigma})(\underset{\sim}{z})| \equiv |\underset{\sim}{k}(\underset{\sim}{h}_1,\underline{\Sigma})(\underset{\sim}{z}) - \underset{\sim}{k}(\underset{\sim}{h}_2,\underline{\Sigma})(\underset{\sim}{z})|$$

$$\leq \int_{B_\gamma} \frac{\left|\frac{\partial\hat{\underset{\sim}{j}}}{\partial\underset{\sim}{h}}(t\underset{\sim}{h}_1(\underset{\sim}{x}) + (1 - t)\underset{\sim}{h}_2(\underset{\sim}{x}),\underline{\Sigma}(x),x)\right||\delta\underset{\sim}{h}(x)|}{|\underset{\sim}{z} - \underset{\sim}{x}|^2} dv(\underset{\sim}{x})$$

where $t \in [0,1]$. We now identify $f(\underset{\sim}{x})$ with the numerator of the integrand in the right-most term of (5.8). We henceforth suppress the arguments of the functions appearing in this numerator. Let us choose β and ν as above, noting that $\beta < \alpha$. From (5.7), (5.8), and the Hölder inequality we then obtain

$$(5.9) \qquad \|\delta \underset{\sim}{k}, L_\alpha(B_\gamma)\| \le \kappa(\beta,\gamma)\{\int_{B_\gamma} |\partial \hat{\underset{\sim}{j}}/\partial \underset{\sim}{h}|^\beta |\delta \underset{\sim}{h}|^\beta dv\}^{1/\beta}$$

$$\le \kappa(\beta,\gamma)\|\partial \hat{\underset{\sim}{j}}/\partial \underset{\sim}{h}, L_{\alpha\beta/(\alpha-\beta)}(B_\gamma)\|\|\delta \underset{\sim}{h}, L_\alpha(B_\gamma)\| .$$

Since $\alpha\beta/(\alpha - \beta) = \beta$, condition (5.5) implies that $\partial \hat{\underset{\sim}{j}}/\partial \underset{\sim}{h} \in L_{\alpha\beta/(\alpha-\beta)}(B_\gamma)$ so that the rightmost term of (5.9) is well defined.

We now prove the compactness of $\underline{\Delta} \mapsto \hat{\underset{\sim}{k}}(\underline{\Delta},\cdot)$. The compactness of K introduced in (5.7b) implies that the mapping

$$(5.10) \qquad L_\beta(B_\gamma) \ni \underset{\sim}{j} \mapsto \left[z \mapsto \int_{B_\gamma} \frac{\underset{\sim}{j}(\underset{\sim}{x}) \wedge (\underset{\sim}{x} - \underset{\sim}{z})}{|\underset{\sim}{x} - \underset{\sim}{z}|^3} dv(\underset{\sim}{z})\right] \in L_\alpha(B_\gamma)$$

is compact. Condition (5.4) ensures that $(\underset{\sim}{h},\underline{\Sigma}) \mapsto \hat{\underset{\sim}{j}}(\underset{\sim}{h},\underline{\Sigma},\cdot)$ takes $L_\alpha(B_\gamma)$ to $L_\beta(B_\gamma)$. By the properties of Nemytskii operators (cf. Krasnosel'skii (1956, Sec. I.2), this mapping is continuous. Since

$$L_\alpha(B_\gamma) \ni \underline{\Delta} \mapsto \int_{B_\gamma} \frac{\hat{\underset{\sim}{j}}(\partial\psi(\underset{\sim}{x})/\partial \underset{\sim}{z} + \hat{\underset{\sim}{k}}(\underline{\Delta},\underset{\sim}{x}),\underline{\Sigma}(\underset{\sim}{x})\underset{\sim}{x}) \wedge (\underset{\sim}{x} - \underset{\sim}{z})dv(\underset{\sim}{x})}{|\underset{\sim}{x} - \underset{\sim}{z}|^3} \equiv \hat{\underset{\sim}{k}}(\underline{\Delta},\underset{\sim}{z})$$

is the composition of a compact with continuous operators, it is compact. □

It follows from (5.5) and the properties of κ that we can make the coefficient of $\|\delta h, L_\alpha\|$ in the right most term of (5.9) less than unity by fixing θ and taking γ small enough or by fixing γ and taking θ small enough or by taking each small enough. The Contraction Mapping Principle ensures that (5.2) has a unique solution giving $\underset{\sim}{h}$ as a continuous function of $\underline{\Delta}$. The composite function obtained by substituting this solution $\underset{\sim}{h}$ into $\underset{\sim}{k}(\underset{\sim}{h},\underline{\Sigma},\cdot)$ has value denoted by $\hat{\underset{\sim}{k}}(\underline{\Delta},\underset{\sim}{z})$.

A number of related results, including some for unbounded domains, can be based on the techniques presented by Sobolev (1950, §§6,9), Stein (1970, Ch. V),

and Kantorovich & Akilov (1977, Chap. XI). Note that Theorem 5.4 says that (5.6) is valid provided the dependence of $\hat{\underset{\sim}{j}}$ on $\underset{\sim}{h}$ becomes weaker as the body $\mathcal{B}$ becomes larger.

We now suppose that $\hat{\underset{\sim}{j}}$ is such that $\underset{\sim}{h}$ admits a representation of the form (5.6). We substitute (5.6) into the right side of (4.2) to get

(5.11) $\underset{\sim}{T}(\underset{\sim}{z}) = \hat{\underset{\sim}{T}}(\underset{\sim}{F}(\underset{\sim}{z}), \underset{\sim}{g}(\underset{\sim}{z}), \lambda(\underset{\sim}{z}), \partial\varphi(\underset{\sim}{z})/\partial\underset{\sim}{z}, \partial\psi(\underset{\sim}{z})/\partial\underset{\sim}{z} + \hat{\underset{\sim}{k}}(\underline{\Delta}, \underset{\sim}{z}), \underset{\sim}{z})$, etc.

Our <u>governing equations</u> are obtained by substituting (5.11) into (3.5)-(3.8):

(5.12) $$\mathrm{Div}\,\hat{\underset{\sim}{T}} = \underset{\sim}{0}\ ,$$

(5.13) $$\mathrm{Div}\,\hat{\underset{\sim}{q}} + \hat{\underset{\sim}{j}}\cdot(\partial\varphi/\partial\underset{\sim}{z}) = 0\ ,$$

(5.14) $$\mathrm{Div}\,\hat{\underset{\sim}{d}} = \sigma\ ,$$

(5.15) $$\mathrm{Div}\,\hat{\underset{\sim}{b}} = 0\ ,$$

where the arguments of the constitutive functions, decorated with carets, are indicated in (5.11). Equations (5.12)-(5.15), having six scalar components, form *a quasilinear system of partial functional differential equations* for the six unknown components of $\underset{\sim}{\chi}, \lambda, \varphi, \psi$. All other variables we have introduced can be expressed in terms of these.

Toupin (1956) took the polarization $\tilde{\underset{\sim}{p}}$ and magnetization $\tilde{\underset{\sim}{m}}$ as independent constitutive variables. One of the goals of our paper is to exhibit the mathematical advantages of choosing $\underset{\sim}{d}, \underset{\sim}{b}, \underset{\sim}{j}$ to be dependent constitutive variables and choosing $\underset{\sim}{e}$ and $\underset{\sim}{h}$ to be independent constitutive variables. In this regard we generalize formulations of Pao & Hutter (1978), Jordan & Eringen (1964), and Ersoy & Kiral (1978) inter alia.

6. Ellipticity and Growth Conditions

Our basic constitutive assumptions are expressed in terms of the quadratic form

(6.1) $\omega(\mathbf{A},\mathbf{t},\mathbf{u},\mathbf{v}) \equiv$

$$\begin{aligned}&\mathbf{A}:(\partial\hat{\mathbf{T}}/\partial\mathbf{F}):\mathbf{A} + \mathbf{A}:(\partial\hat{\mathbf{T}}/\partial\mathbf{g})\cdot\mathbf{t} + \mathbf{A}:(\partial\hat{\mathbf{T}}/\partial\mathbf{e})\cdot\mathbf{u} + \mathbf{A}:(\partial\hat{\mathbf{T}}/\partial\mathbf{h})\cdot\mathbf{v}\\ &+ \mathbf{t}\cdot(\partial\hat{\mathbf{q}}/\partial\mathbf{F}):\mathbf{A} + \mathbf{t}\cdot(\partial\hat{\mathbf{q}}/\partial\mathbf{g})\cdot\mathbf{t} + \mathbf{t}\cdot(\partial\hat{\mathbf{q}}/\partial\mathbf{e})\cdot\mathbf{u} + \mathbf{t}\cdot(\partial\hat{\mathbf{q}}/\partial\mathbf{h})\cdot\mathbf{v}\\ &+ \mathbf{u}\cdot(\partial\hat{\mathbf{d}}/\partial\mathbf{F}):\mathbf{A} + \mathbf{u}\cdot(\partial\hat{\mathbf{d}}/\partial\mathbf{g})\cdot\mathbf{t} + \mathbf{u}\cdot(\partial\hat{\mathbf{d}}/\partial\mathbf{e})\cdot\mathbf{u} + \mathbf{u}\cdot(\partial\hat{\mathbf{d}}/\partial\mathbf{h})\cdot\mathbf{v}\\ &+ \mathbf{v}\cdot(\partial\hat{\mathbf{b}}/\partial\mathbf{F}):\mathbf{A} + \mathbf{v}\cdot(\partial\hat{\mathbf{b}}/\partial\mathbf{g})\cdot\mathbf{t} + \mathbf{v}\cdot(\partial\hat{\mathbf{b}}/\partial\mathbf{e})\cdot\mathbf{u} + \mathbf{v}\cdot(\partial\hat{\mathbf{b}}/\partial\mathbf{h})\cdot\mathbf{v}\ .\end{aligned}$$

If $\omega(\mathbf{A},\mathbf{t},\mathbf{u},\mathbf{v}) > 0\ \forall(\mathbf{A},\mathbf{t},\mathbf{u},\mathbf{v}) \neq (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$, then $(\hat{\mathbf{T}},\hat{\mathbf{q}},\hat{\mathbf{d}},\hat{\mathbf{b}})$ is said to be <u>strictly monotone</u>. The use of this attractive mathematical restriction would deprive the theory of much of its physical versatility. Among its adverse consequences (discussed in detail by Antman (1983)) is that the uniqueness theorems it implies effectively prevent the buckling of a column of such a material however slender under a compressive load however large.

We can eliminate this kind of uniqueness in the mechanical response while preserving it fully in electromagnetic response and partially in the thermal response by weakening the strict monotonicity condition. If $\omega(\mathbf{r}\mathbf{s},\mathbf{t},\mathbf{u},\mathbf{v}) > 0$ $\forall(\mathbf{r},\mathbf{s},\mathbf{t},\mathbf{u},\mathbf{v}) \neq (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0})$, then we say that $(\hat{\mathbf{T}},\hat{\mathbf{q}},\hat{\mathbf{d}},\hat{\mathbf{b}})$ satisfies the (strict form of the) <u>restricted strong ellipticity condition</u>. (Note that $\mathbf{A}$ equals the dyadic product $\mathbf{r}\mathbf{s}$ if and only if $\mathbf{A}$ has rank 1.) In Section 8b we discuss the physical significance of the restricted strong ellipticity condition. If $\omega(\mathbf{r}\mathbf{s},\xi_1\mathbf{s},\xi_2\mathbf{s},\xi_3\mathbf{s}) > 0\ \forall(\mathbf{r},\mathbf{s},\xi_1,\xi_2,\xi_3) \neq (\mathbf{0},\mathbf{0},0,0,0)$, then $(\hat{\mathbf{T}},\hat{\mathbf{q}},\hat{\mathbf{d}},\hat{\mathbf{b}})$ satisfies the (strict form of the) <u>strong ellipticity condition</u>. This condition is the generalization to elliptic systems in divergence form of the Legendre-Hadamard condition of the calculus of variations.

In this paper we shall study the strong ellipticity condition and its restricted form. Our subject is insufficiently developed to determine whether

phenomena permitted by the strong ellipticity condition, but prohibited by its restricted form, are observed (cf. Sec. 8b). The intuitive content of these conditions is described in Section 1. Most special theories of material behavior of electro-magneto-thermo-elasticity satisfy the restricted strong ellipticity condition because many of the "off-diagonal" terms in (6.1) are zero. (But recent work on the study of plastic effects and phase changes treates theories of elastic solids for which even $\mathbf{rs}:(\partial\hat{\mathbf{T}}/\partial\mathbf{F}):\mathbf{rs}$ need not be positive for $\mathbf{rs} \neq \mathbf{O}$. Cf. Ericksen (1980).)

We now study the behavior of the constitutive equations at extreme values of their arguments. The conditions we impose must be consistent with the strong ellipticity condition. Since our work is just a generalization of that of Antman (1983), we omit an extensive commentary. In Sections 8 and 11 we describe more specific conditions appropriate for special problems.

Recall that

$$\underline{\Gamma} \equiv (\mathbf{F},\mathbf{g},\lambda,\mathbf{e},\mathbf{h}) \;. \tag{6.2}$$

Let $\mathbf{a}$ and $\mathbf{c}$ be unit vector fields depending on $\underline{\Gamma},\mathbf{y},\mathbf{z}$. The strong ellipticity condition implies that

$$\frac{\partial(\mathbf{a}\cdot\hat{\mathbf{T}}\cdot\mathbf{c})}{\partial(\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{c})} > 0 \quad \text{if } \mathbf{a} \text{ and } \mathbf{c} \text{ are independent of } \mathbf{a}\cdot\mathbf{F}\cdot\mathbf{c} \,, \tag{6.3a}$$

$$\frac{\partial(\hat{\mathbf{q}}\cdot\mathbf{a})}{\partial(\mathbf{g}\cdot\mathbf{a})} > 0 \quad \text{if } \mathbf{a} \text{ is independent of } \mathbf{g}\cdot\mathbf{a} \,, \tag{6.3b}$$

$$\frac{\partial(\hat{\mathbf{d}}\cdot\mathbf{a})}{\partial(\mathbf{e}\cdot\mathbf{a})} > 0 \quad \text{if } \mathbf{a} \text{ is independent of } \mathbf{e}\cdot\mathbf{a} \,, \tag{6.3c}$$

$$\frac{\partial(\hat{\mathbf{b}}\cdot\mathbf{a})}{\partial(\mathbf{h}\cdot\mathbf{a})} > 0 \quad \text{if } \mathbf{a} \text{ is independent of } \mathbf{h}\cdot\mathbf{a} \,. \tag{6.3d}$$

Moreover, if $\mathbf{a}$ and $\mathbf{c}$ are independent of $\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{c}$, then

(6.4) $$\mathcal{D}(\mathbf{ac}) \equiv \{\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{c} \in \mathbb{R} : \det \mathbf{F} > 0\}$$

is either an open half-line, or the whole line, or empty and can then be written as

(6.5) $$\mathcal{D}(\mathbf{ac}) = (\ell^{-}(\mathbf{ac}), \ell^{+}(\mathbf{ac})) .$$

We suppress the dependence of $\mathcal{D}$ and $\ell^{\pm}$ on Γ, χ, z. The facts motivate the following

<u>6.6. Hypothesis. Let $\mathbf{a}$ and $\mathbf{c}$ be unit vector fields depending on Γ, χ, z. If $\mathcal{D}(\mathbf{ac})$ is an open half-line or the whole line and if $\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{c} \mapsto \mathbf{a}\cdot\hat{\mathbf{T}}\cdot\mathbf{c}$ is strictly increasing, then</u>

(6.7a) $\mathbf{a}\cdot\hat{\mathbf{T}}\cdot\mathbf{c} \to \pm\infty$ <u>as</u> $\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{c} \to \ell^{\pm}(\mathbf{ac})$ <u>for fixed</u> $\Gamma - ((\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{c})\mathbf{ac}, \mathbf{0}, 0, \mathbf{0}, \mathbf{0}), \chi, z$.

<u>If</u> $\mathbf{g}\cdot\mathbf{a} \mapsto \hat{\mathbf{g}}\cdot\mathbf{a}$ <u>is strictly increasing, then</u>

(6.7b) $\hat{\mathbf{g}}\cdot\mathbf{a} \to \pm\infty$ <u>as</u> $\mathbf{g}\cdot\mathbf{a} \to \pm\infty$ <u>for fixed</u> $\Gamma - (\mathbf{0}, (\mathbf{g}\cdot\mathbf{a})\mathbf{a}, 0, \mathbf{0}, \mathbf{0}), \chi, z$.

<u>If</u> $\mathbf{e}\cdot\mathbf{a} \mapsto \hat{\mathbf{d}}\cdot\mathbf{a}$ <u>is strictly increasing, then</u>

(6.7c) $\hat{\mathbf{d}}\cdot\mathbf{a} \to \pm\infty$ <u>as</u> $\mathbf{e}\cdot\mathbf{a} \to \pm\infty$ <u>for fixed</u> $\Gamma - (\mathbf{0}, \mathbf{0}, 0, (\mathbf{e}\cdot\mathbf{a})\mathbf{a}, \mathbf{0}), \chi, z$.

<u>If</u> $\mathbf{h}\cdot\mathbf{a} \mapsto \hat{\mathbf{b}}\cdot\mathbf{a}$ <u>is strictly increasing, then</u>

(6.7d) $\hat{\mathbf{b}}\cdot\mathbf{a} \to \pm\infty$ <u>as</u> $\mathbf{h}\cdot\mathbf{a} \to \pm\infty$ <u>for fixed</u> $\Gamma - (\mathbf{0}, \mathbf{0}, 0, \mathbf{0}, (\mathbf{h}\cdot\mathbf{a})\mathbf{a}), \chi, z$.

If $\mathcal{D}(\mathbf{ac})$ is a half-line, which happens exactly when the cofactor of $\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{c}$ in $\det \mathbf{F}$ does not vanish, then $\partial\mathcal{D}(\mathbf{ac})$ is a point (either $\ell^{+}(\mathbf{ac})$ or $\ell^{-}(\mathbf{ac})$). Then (6.7a) implies that we can define a function

(6.8) $$\mathbf{ac} \mapsto \delta(\mathbf{ac}) \in \{-1, 1\}$$

such that

(6.9) $$\delta(\mathbf{ac})\mathbf{a}\cdot\hat{\mathbf{T}}\cdot\mathbf{c} \to -\infty \quad \text{as} \quad \mathbf{a}\cdot\mathbf{F}\cdot\mathbf{c} \to \partial\mathcal{D}(\mathbf{ac})$$

for fixed $\Gamma - ((\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{c})\mathbf{ac}, \mathbf{0}, 0, \mathbf{0}, \mathbf{0}), \chi, z$. Note that $\det \mathbf{F} \to 0$ so the local volume ratio shrinks to 0 as $\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{c} \to \partial\mathcal{D}(\mathbf{ac})$.

We now complement Hypothesis 6.6 in a way that promotes the analysis of Part II by describing the behavior of the constitutive function $\hat{\mathbf{T}}$ as more than one component of Γ are allowed to vary.

6.10. Hypothesis. Let $\underset{\sim}{z}$ be fixed. Let $\{\underset{\sim}{E}_\tau, \tau = 1,\ldots,9\}$ be a basis for Lin consisting of dyadic products of unit vectors, which may depend on $\underline{\Gamma}, \underset{\sim}{y}, \underset{\sim}{z}$, and let $\{\underset{\sim}{E}_\tau^*\}$ be the basis dual to $\{\underset{\sim}{E}_\tau\}$. Let $\underset{\sim}{F}:\underset{\sim}{E}_\tau \mapsto \underset{\sim}{T}:\underset{\sim}{E}_\tau^*$ be strictly increasing for each τ. Let $\{\underline{\Gamma}_n\}$ be a sequence of states such that the $\ell^\pm(\underset{\sim}{E}_\tau)$ formed from $\{\underline{\Gamma}_n\}$ are actually independent of n. Let the set of integers $\{1,\ldots,9\}$ be written as a disjoint union $a \cup b \cup c \cup d \cup e \cup f$ with

(6.11a) $\partial\mathcal{D}(\underset{\sim}{E}_\tau) \neq \emptyset$ and $\delta(\underset{\sim}{E}_\tau)\hat{\underset{\sim}{T}}(\underline{\Gamma}_n,\underset{\sim}{z}):\underset{\sim}{E}_\tau^* \to -\infty$ for $\tau \in a$,

(6.11b) $\partial\mathcal{D}(\underset{\sim}{E}_\tau) \neq \emptyset$ and $\underset{\sim}{F}_n:\underset{\sim}{E}_\tau \to \partial\mathcal{D}(\underset{\sim}{E}_\tau)$ for $\tau \in b$,

(6.11c) $\partial\mathcal{D}(\underset{\sim}{E}_\tau) \neq \emptyset$ and $\underset{\sim}{F}_n:\underset{\sim}{E}_\tau \in$ compact subset of $(\ell^-(\underset{\sim}{E}_\tau), \ell^+(\underset{\sim}{E}_\tau))$ for $\tau \in c$,

(6.11d) $\partial\mathcal{D}(\underset{\sim}{E}_\tau) \neq \emptyset$ for $\tau \in d$,

(6.11e) $\partial\mathcal{D}(\underset{\sim}{E}_\tau) = \emptyset$ and $\underset{\sim}{F}_n:\underset{\sim}{E}_\tau \in$ compact subset of $(\ell^-(\underset{\sim}{E}_\tau), \ell^+(\underset{\sim}{E}_\tau))$ for $\tau \in e$,

(6.11f) $\partial\mathcal{D}(\underset{\sim}{E}_\tau) = \emptyset$ for $\tau \in f$.

If $a \neq \emptyset$, then

(6.12a) $\delta(\underset{\sim}{E}_\tau)\hat{\underset{\sim}{T}}(\underline{\Gamma}_n,z):\underset{\sim}{E}_\tau^* \to -\infty \ \forall\tau \in a \cup b \cup c$; for each τ in d either

$|\underset{\sim}{F}_n:\underset{\sim}{E}_\tau| \to \infty$ or $\delta(\underset{\sim}{E}_\tau)\hat{\underset{\sim}{T}}(\underline{\Gamma}_n,\underset{\sim}{z}):\underset{\sim}{E}_\tau^* \to -\infty$.

If $b \neq \emptyset$, then

(6.12b) either (i) $\delta(E_\tau)\hat{\underset{\sim}{T}}(\underline{\Gamma}_n,\underset{\sim}{z}):\underset{\sim}{E}_\tau^* \to -\infty \ \forall\tau \in a \cup d \cup c \cup d$ or else

(ii) $\exists\tau \in a \cup d$ such that $|\underset{\sim}{F}_n:\underset{\sim}{E}_\tau| \to \infty$ and $\exists\tau \in a \cup b \cup c \cup d$ such that $|\underset{\sim}{T}(\underline{\Gamma}_n,\underset{\sim}{z}):\underset{\sim}{E}_\tau^*| \to \infty$.

Moreover, the dualization obtained by respectively replacing the statements

$$\underset{\sim}{F}_n:\underset{\sim}{E}_\tau \to \partial\mathcal{D},(\underset{\sim}{E}_\tau), |\underset{\sim}{F}_n:\underset{\sim}{E}_\tau| \to \infty,\ \delta\hat{\underset{\sim}{T}}(\underline{\Gamma}_n,\underset{\sim}{z}):\underset{\sim}{E}_\tau^* \to -\infty$$

appearing in (6.11), (6.12) by their opposites

$$|\underset{\sim}{F}_n:\underset{\sim}{E}_\tau| \to \infty,\ \underset{\sim}{F}_n:\underset{\sim}{E}_\tau \to \partial\mathcal{D}(\underset{\sim}{E}_\tau),\ \delta\hat{\underset{\sim}{T}}(\underline{\Gamma}_n,\underset{\sim}{z}):\underset{\sim}{E}_\tau^* \to \infty$$

is also valid.

The statement containing (6.12b) may be loosely interpreted thus: If there are fibers compressed to zero length in some directions, then either the material is squeezed out with an infinite stretch in another direction or else it is

prevented from doing so by infinite compressive stresses in all other directions. The other statements have similar interpretations. The whole hypothesis effectively says that extreme behavior in one direction must be accompanied by extreme behavior in some other direction. This observation is used in Part II to establish regularity results by showing that behavior could be extreme in only one direction and therefore cannot be extreme. It would be easy to generalize Hypothesis 6.7 to account for extreme couplings between the mechanical, electromagnetic, and thermal effects, but the intuitive evidence for such a generalization is not compelling.

It is important to note that the transformations (3.3) and (3.4) ensure that the Maxwell stress contains terms with $(\det \underset{\sim}{F})^{-1}$ as factors. These terms could compete with the "purely mechanical part of this stress" when $\det \underset{\sim}{F}$ is small. Our constitutive hypotheses on the effective stress control this competition. They say that the material response in large compression is dominated by that for purely mechanical response.

7. Boundary Conditions. The Principle of Virtual Work

At a boundary point $\mathbf{z} \in \partial\mathcal{B}$ we may prescribe the position $\mathbf{y}(\mathbf{z})$ or merely subject it to certain constraints, such as the requirement that it be *confined* to a fixed surface. To account for the varied possibilities it is convenient to describe such boundary conditions in the language of holonomic constraints. We accordingly specify

$$\mathbf{y}(\mathbf{z}) = \bar{\mathbf{y}}(\mathbf{z},\underline{r}) \quad \text{for each} \quad \mathbf{z} \in \partial\mathcal{B} \tag{7.1}$$

where $\bar{\mathbf{y}}$ is a given function continuously differentiable in $\underline{r} \in \mathbb{R}^3$, which represents the set of generalized coordinates for $\mathbf{y}(\mathbf{z})$. The rank of $\partial\bar{\mathbf{y}}/\partial\underline{r}$ is the number of degrees of freedom of $\mathbf{z}$. Equation (7.1) restricts $\mathbf{y}(\mathbf{z})$ to a manifold. The set of vectors $\mathbf{y}^{\#}(\mathbf{z})$ of the form $[\partial\bar{\mathbf{y}}(\mathbf{z},\underline{r})/\partial\underline{r}]\cdot\underline{r}^{\#}(\mathbf{z})$ for $\underline{r}^{\#}(\mathbf{z}) \in \mathbb{R}^3$ form the tangent space to this manifold at $\mathbf{y}(\mathbf{z})$. The elements $\mathbf{y}^{\#}(\mathbf{z})$ of this tangent space are called virtual displacements. We complement (7.1) by specifying the projection of the traction $\mathbf{T}\cdot\mathbf{n}$ on this tangent space:

$$(\mathbf{n}\cdot\mathbf{T}^{*} - \bar{\mathbf{t}})\cdot(\partial\bar{\mathbf{y}}/\partial\underline{r}) = \underline{0} \quad \text{at each} \quad \mathbf{z} \in \partial\mathcal{B} \tag{7.2}$$

where $\bar{\mathbf{t}}$ is a given function of $\mathbf{y}'\mathbf{z})$, $\lambda(\mathbf{z})$, $\varphi(\mathbf{z})$, $\psi(\mathbf{z})$, $\mathbf{z}$ and possibly other variables. Thus

$$(\mathbf{n}\cdot\mathbf{T}^{*} - \bar{\mathbf{t}})\cdot\mathbf{y}^{\#} = 0 \quad \text{on} \quad \partial\mathcal{B} \,. \tag{7.3}$$

At each $\mathbf{z} \in \partial\mathcal{B}$ we also prescribe

(7.4a,b) either $\lambda(\mathbf{z}) = \bar{\lambda}(\mathbf{z})$ or $\mathbf{g}(\mathbf{z})\cdot\mathbf{n}(\mathbf{z}) = \bar{\gamma}(\mathbf{y}(\mathbf{z}),\lambda(\mathbf{z}),\varphi(\mathbf{z}),\psi(\mathbf{z}),\mathbf{z})$,

(7.5a,b) either $\varphi(\mathbf{z}) = \bar{\varphi}(\mathbf{z})$ or $\mathbf{d}(z)\cdot\mathbf{n}(z) = \bar{\delta}(\mathbf{y}(\mathbf{z}),\lambda(\mathbf{z}),\varphi(\mathbf{z}),\psi(\mathbf{z}),\mathbf{z})$,

(7.6a,b) either $\psi(\mathbf{z}) = \bar{\psi}(\mathbf{z})$ or $\mathbf{b}(z)\cdot\mathbf{n}(z) = \bar{\beta}(\mathbf{y}(\mathbf{z}),\lambda(\mathbf{z}),\varphi(\mathbf{z}),\psi(\mathbf{z}),\mathbf{z})$.

Let $\lambda^{\#},\varphi^{\#},\psi^{\#}$ be arbitrary continuous functions on $\partial\mathcal{B}$ that respectively vanish where (7.4a), (7.5a), (7.6a) hold. They are virtual fields. Then in analogy with (7.3) we have

$$(\mathbf{g}\cdot\mathbf{n} - \bar{\gamma})\lambda^{\#} + (\mathbf{d}\cdot\mathbf{n} - \bar{\delta})\varphi^{\#} + (\mathbf{b}\cdot\mathbf{n} - \bar{\beta})\psi^{\#} = 0 \quad \text{on} \quad \partial\mathcal{B} \,. \tag{7.7}$$

Our fundamental equations of balance are the integral versions of (3.5)-(3.7), which are to hold over "almost all" nice subbodies of $\mathcal{B}$. These equations can be supplemented by appropriately weakened forms of the boundary conditions we have just listed. Antman & Osborn (1979) show (strictly speaking, for the purely mechanical problem) that when all the integrals make sense as Lebesgue integrals, then these equations and boundary conditions are equivalent to the <u>Principle of Virtual Work</u>

$$\begin{aligned}(7.8)\qquad & \int_{\mathcal{B}} [\hat{\underset{\sim}{T}} : (\partial \underset{\sim}{\chi}^{\#}/\partial \underset{\sim}{z}) + \hat{\underset{\sim}{g}}\cdot(\partial\lambda^{\#}/\partial \underset{\sim}{z}) + \hat{\underset{\sim}{d}}\cdot(\partial\varphi^{\#}/\partial \underset{\sim}{z}) + \hat{\underset{\sim}{b}}\cdot(\partial\psi^{\#}/\partial \underset{\sim}{z})]dv \\ & - \int_{\mathcal{B}} (\underset{\sim}{f}\cdot\underset{\sim}{\chi}^{\#} + \hat{\rho}\lambda^{\#} - \sigma\varphi^{\#})dv \\ & = \int_{\partial\mathcal{B}} (\bar{\underset{\sim}{t}}\cdot\underset{\sim}{\chi}^{\#} + \bar{\gamma}\lambda^{\#} + \bar{\delta}\varphi^{\#} + \bar{\beta}\psi^{\#})da\end{aligned}$$

for all reasonably nice fields $\underset{\sim}{\chi}^{\#}, \lambda^{\#}, \varphi^{\#}, \psi^{\#}$ having the boundary behavior specified above. Equation (7.8) is just the weak formulation of our boundary value problem consisting of (5.12)-(5.15) subject to (7.1), (7.2), (7.4)-(7.6). The arguments of $\hat{\underset{\sim}{T}}$, etc., are given in (5.11).

In many circumstances the deformation of a body subjected to the action of external electromagnetic fields changes the ambient fields. Thus there would be a complete coupling between the fields interior and exterior to the body. Since our goal is to study the role of the constitutive assumptions of Section 6, we are avoiding such coupled problem by restricting electromagnetic boundary conditions to (8.5) and (8.6). Methods for treating fully coupled problems would be similar to those of Section 5.

8. General Existence Theorems

In this section we obtain existence theorems for two important special classes of problems, which can be readily treated by means of recent results for elliptic systems. For the first problem we assume that there is neither thermal nor electrical conduction and that there is a stored energy function; thus this reduced problem admits a variational formulation. For the second problem we assume that the material is rigid. The restricted strong ellipticity condition then reduces to a monotonicity condition, which is capable of handling our nonlocal operators.

a. Conservative Problems

We assume that the material does not conduct electricity so that the constitutive function $\hat{\underset{\sim}{j}} = \underset{\sim}{0}$. Thus the Joule heating is zero (cf. (5.13)). Moreover, (5.2) reduces to

$$\underset{\sim}{h} = \partial\psi/\partial\underset{\sim}{z} \ . \tag{8.1}$$

We assume that one of the following conditions holds:

i) $\hat{\underset{\sim}{q}}, \hat{\rho}, \bar{\gamma}$ depend only on $\underset{\sim}{g}, \lambda, \underset{\sim}{z}$ and the boundary value problem (5.13), (7.4) has a weak solution λ in a suitable Sobolev space. (In part (b) below, we show how a slight strengthening of our hypotheses ensures the existence of λ.)

ii) The boundary value problem (5.13), (7.4) has a solution λ independent of the fields $\underset{\sim}{F}$, $\partial\varphi/\partial\underset{\sim}{z}$, $\partial\psi/\partial\underset{\sim}{z}$. This situation would occur if $\hat{\underset{\sim}{q}} = \underset{\sim}{0}$ when $\underset{\sim}{g} = \underset{\sim}{0}$ and if λ is prescribed to be constant λ_0 on $\partial\mathcal{B}$, for then the boundary value problem would admit a solution $\lambda = \lambda_0$ on all of $\mathcal{B}$.

iii) The constitutive functions $\hat{\underset{\sim}{T}}, \hat{\underset{\sim}{d}}, \hat{\underset{\sim}{b}}$ are independent of $\underset{\sim}{g}$ and λ.

In cases (i) and (ii) we can substitute the solution λ and its gradient $\underset{\sim}{g}$ into (5.12), (5.14), (5.15), (7.1), (7.2), (7.5), (7.6). Since λ is known,

its presence in these equations merely changes the dependence of the constitutive functions on $\underset{\sim}{z}$. In case (iii), these equations are unaffected by the solution λ.

We assume that σ is a prescribed function of $\underset{\sim}{z}$. We finally assume that there is a stored energy function W depending on $\underset{\sim}{F}, \underset{\sim}{g}, \lambda, \underset{\sim}{e}, \underset{\sim}{h}, \underset{\sim}{z}$ with W continuously differentiable in $\underset{\sim}{F}, \underset{\sim}{e}, \underset{\sim}{h}$, continuous in $\underset{\sim}{g}$ and λ, and measurable in $\underset{\sim}{z}$, for all values of the remaining arguments, such that

(8.2) $$\hat{\underset{\sim}{T}} = \partial W/\partial \underset{\sim}{F}, \quad \hat{\underset{\sim}{d}} = \partial W/\partial \underset{\sim}{e}, \quad \hat{\underset{\sim}{b}} = \partial W/\partial \underset{\sim}{h} .$$

(The Clausius-Duhem inequality would deliver a specific thermodynamic function for W and show that it would be independent of $\underset{\sim}{g}$.) The discussion following assumptions (i), (ii), (iii) motivates us to suppress the dependence of W on $\underset{\sim}{g}$ and λ, their effects being absorbed by the dependence of W on $\underset{\sim}{z}$.

We suppose that the body force $\underset{\sim}{f}$ is conservative so that there is a function $(\underset{\sim}{y}, \underset{\sim}{z}) \mapsto U(\underset{\sim}{y}, \underset{\sim}{z})$, with $U(\cdot, \underset{\sim}{z})$ continuously differentiable for all $\underset{\sim}{z}$ in $\mathcal{B}$ and with $U(\underset{\sim}{y}, \cdot)$ measurable for all $\underset{\sim}{y} \in \mathbb{E}^3$, such that

(8.3) $$\underset{\sim}{f} = -\partial U/\partial \underset{\sim}{y} .$$

We suppose that $\bar{\underset{\sim}{\tau}}, \bar{\delta}, \bar{\beta}$ of (7.2), (7.5), (7.6) are conservative so that there is a function

$$\mathbb{E}^3 \times \mathbb{E}^3 \times \mathbb{E}^3 \times \partial\mathcal{B} \ni (\underset{\sim}{y}, \underset{\sim}{e}, \underset{\sim}{h}, \underset{\sim}{z}) \mapsto V(\underset{\sim}{y}, \underset{\sim}{e}, \underset{\sim}{h}, \underset{\sim}{z})$$

with $V(\cdot,\cdot,\cdot,\underset{\sim}{z})$ continuously differentiable for all $\underset{\sim}{z}$ in $\partial\mathcal{B}$ and with $V(\underset{\sim}{y}, \underset{\sim}{e}, \underset{\sim}{h}, \cdot)$ measurable for all $\underset{\sim}{y}, \underset{\sim}{e}, \underset{\sim}{h}$, such that

(8.4) $$\bar{\underset{\sim}{\tau}} = \partial V/\partial \underset{\sim}{y}, \quad \bar{\delta} = \partial V/\partial \underset{\sim}{e}, \quad \bar{\beta} = \partial V/\partial \underset{\sim}{h} .$$

(The domain of $V(\underset{\sim}{y}, \underset{\sim}{e}, \underset{\sim}{h}, \cdot)$ may be taken to be the closure of $\partial\mathcal{B} \backslash \{\underset{\sim}{z} \in \partial\mathcal{B} : \underset{\sim}{\chi}^{\#}(\underset{\sim}{z}) = \underset{\sim}{0}, \ \varphi^{\#}(\underset{\sim}{z}) = 0, \ \psi^{\#}(\underset{\sim}{z}) = 0\}$. (See Section 7.) V could conceivably depend on $\underset{\sim}{g}$ and λ. We suppress any such dependence in accord with the policy we have adopted above.

We finally assume that $\partial\mathcal{B}$ is bounded and has a locally Lipschitz continuous graph. Moreover, we require that the supports of $\varphi^{\#}, \psi^{\#}$, and the components of $\chi^{\#}$ be nice enough to ensure that the boundary conditions (7.1), (7.5a), (7.6a) are assumed in the sense of trace when χ, φ, ψ lie in Sobolev spaces of the form $W_p^1(B)$ with $p > 1$. (Necessary conditions for these properties are not known. See the discussion of Antman & Osborn (1979).)

Under these conditions the weak form of the Euler-Lagrange equations for the functional

$$(8.5) \qquad I(\chi,\varphi,\psi) = \int_B \left[W\left(\frac{\partial\chi}{\partial z}(z), \frac{\partial\varphi}{\partial z}(z), \frac{\partial\psi}{\partial z}(z), z\right) + U(\chi(z), z) + \sigma(z)\varphi(z)\right]dv(z)$$

$$+ \int_{\partial B} V(\chi(z), \varphi(z), \psi(z), z)\,da(z)$$

for χ, φ, ψ satisfying (7.1), (7.5a), (7.6a) have exactly the form of (7.8) with $\lambda^{\#} = 0$. (Of course, many authors take a variational principle, such as this, as the starting point for the derivation of the governing equation for electromechanical interactions. See, e.g., Toupin (1956), Brown (1966), Nelson (1979).)

Let $F^{\times}$ denote the cofactor tensor of F. W is said to be <u>polyconvex</u> (cf. Ball (1977)) if it can be written in the form

$$W(F,e,h,z) = \Omega(F,F^{\times},\det F,e,h,z)$$

with $\Omega(\cdot,\cdot,\cdot,\cdot,\cdot,z)$ convex on $\mathrm{Lin} \times \mathrm{Lin} \times (0,\infty) \times \mathbb{E}^3 \times \mathbb{E}^3$ for each $z \in \mathcal{B}$. The work of Ball (1977) shows that if W is polyconvex, then (10.2) satisfies the restricted strong ellipticity condition of Section 6. To account for (6.7) we require that

$$(8.6) \qquad \Omega(F,F^{\times},\delta,e,h,z) \to \infty \quad \text{as} \quad \delta \to 0 .$$

<u>8.7. Theorem. Let</u> W <u>be polyconvex and satisfy</u> (8.6). <u>Let there be numbers</u> $k > 0$, $p > 2$, $q > p/(p-1)$, $r > 1$ <u>and functions</u> $\omega \in L_1(\mathcal{B})$ <u>and</u>

$\chi \in L_1(\partial\mathcal{B})$ <u>such that</u>

(8.8) $$\Omega(\underset{\sim}{F},\underset{\sim}{F}^\times,\delta,\underset{\sim}{g},\underset{\sim}{h},\underset{\sim}{z}) \geq \omega(\underset{\sim}{z}) + k(|\underset{\sim}{F}|^p + |\underset{\sim}{F}^\times|^q + \delta^r + |\underset{\sim}{g}|^p + |\underset{\sim}{h}|^p)$$

<u>for all</u> $\underset{\sim}{z} \in \mathcal{B}$,

(8.9) $$U(\underset{\sim}{\chi},\underset{\sim}{z}) \geq \omega(\underset{\sim}{z}) \quad \text{for all} \quad \underset{\sim}{z} \in \mathcal{B} \ ,$$

(8.10) $$\sigma \in L_{p/(p-1)}(\mathcal{B}) \ ,$$

(8.11) $$V(\underset{\sim}{\chi},\varphi,\psi,\underset{\sim}{z}) \geq \chi(\underset{\sim}{z}) \quad \text{for all} \quad \underset{\sim}{z} \in \partial\mathcal{B} \ .$$

<u>Let</u> $\partial\mathcal{B}$ <u>have the properties specified above. Let</u>

(8.12) $$\mathcal{W} \equiv \{(\underset{\sim}{\chi},\varphi,\psi) \in W_p^1(\mathcal{B}:\underset{\sim}{F}^\times \in L_q(\mathcal{B}),\ \det \underset{\sim}{F} \in L_r(\mathcal{B}) \ ,$$

$\det \underset{\sim}{F} > 0$ a.e. for $\underset{\sim}{F} = \partial\underset{\sim}{\chi}/\partial\underset{\sim}{z}$; (7.1), (7.5a), (7.6a)

<u>are satisfied in the sense of trace where they are prescribed</u>

<u>on</u> $\partial\mathcal{B}$.}

<u>If there exists an element</u> $(\underset{\sim}{\chi}_1,\varphi_1,\psi_1) \in \mathcal{W}$ <u>such that</u> $I(\underset{\sim}{\chi}_1,\varphi_1,\psi_1) < \infty$, <u>then</u> <u>there exists an element</u> $(\bar{\underset{\sim}{y}},\bar{\varphi},\bar{\psi})$ <u>that minimizes</u> I <u>on</u> $\mathcal{W}$.

The proof of this theorem is effected by making minor adjustments to that of Ball (1977) and is accordingly omitted. (Further developments of Ball's theory, useful for our class of problems, are given by Ball & Murat (1984), Dacorogna (1982), and the references cited therein.)

b. <u>Rigid Conductors</u>

We now study the effects of the conduction of heat and electricity, but confine our attention to rigid bodies, for which $\underset{\sim}{F}$ is constrained to be the identity $\underset{\sim}{I}$. We accordingly take the virtual displacement $\underset{\sim}{\chi}^\#$, appearing in (7.8), to be $\underset{\sim}{0}$. This choice ensures that the First Piola-Kirchhoff stress tensor, which now is the Lagrange multiplier maintaining the constraint of rigidity, does not enter into (7.8). We drop $\underset{\sim}{F}$ from the list of variables constituting $\underset{\sim}{\Delta}$ in (5.1) and from the arguments of the constitutive functions

$\hat{\underset{\sim}{g}}, \hat{\underset{\sim}{d}}, \hat{\underset{\sim}{b}}, \hat{j}$ (cf. (5.11)). Our boundary value problem reduces to (5.13)-(5.15), (7.4)-(7.6), whose weak form is the suitably specialized version of (7.8).

We assume that the restricted strong ellipticity condition holds. Thus $(\underset{\sim}{g}, \underset{\sim}{e}, \underset{\sim}{h}) \mapsto (\hat{\underset{\sim}{g}}(g, \lambda, \underset{\sim}{e}, \underset{\sim}{h}, \underset{\sim}{z}),\ \hat{\underset{\sim}{d}}(g, \lambda, \underset{\sim}{e}, \underset{\sim}{h}, \underset{\sim}{z}),\ \hat{\underset{\sim}{b}}(g, \lambda, \underset{\sim}{e}, \underset{\sim}{h}, \underset{\sim}{z}))$ is strictly monotone. This condition prohibits certain kinds of nonuniqueness.

Since hysteresis frequently is associated with nonuniqueness and since hysteresis is one of the most important phenomena of ferromagnetism, it might appear that our use of the restricted strong ellipticity condition precludes us from dealing with ferromagnetic materials. But molecular theories of ferromagnetism (cf. Tebble (1969)) suggest that hysteresis is associated with constitutive equations with nonlocal effects. If we accept such theories, then to account for ferromagnetism it is necessary to generalize the form of our constitutive functions before relaxing the ellipticity conditions. We do not attempt such a generalization here: Our analysis should be regarded as merely applying to paramagnetic materials. We do, however, examine nonlocal operators that are introduced by the mathematical approach we use to handle electric currents. Some of the methods we use can be applied to more general kinds of nonlocal behavior.

We now outline an existence theory that can be applied directly to our specialized version of (7.8). We first present the theory in an abstract form in order to facilitate a comparison of it with presentations in the mathematical literature. Afterward we make the requisite identifications.

Let $\mathcal{B}$, as before, be the closure of a domain in $\mathbb{R}^3$. We assume that $\partial\mathcal{B}$ has a locally Lipschitz continuous graph. A typical point in $\mathcal{B}$ is denoted $\underset{\sim}{z}$. Let $\underline{u}(\underset{\sim}{z}) = (u_1(\underset{\sim}{z}), \ldots, u_m(\underset{\sim}{z}))$. For $p \in (1,\infty)$, let the operator

(8.13) $$L_p(\mathcal{B})^m \times L_p(\mathcal{B})^{3m} \ni (\underline{u}, \partial\underline{v}/\partial\underset{\sim}{z}) \mapsto \hat{\underline{k}}(\underline{u}, \partial\underline{v}/\partial\underset{\sim}{z})(\cdot) \in [L_p(\mathcal{B})]^r$$

take bounded sets into bounded sets. Let

(8.14) $$\mathbb{R}^m \times \mathbb{R}^{3m} \times \mathbb{R}^r \times \mathcal{B} \ni (\underline{\xi}, \underline{\eta}, \underline{\zeta}, \underset{\sim}{z}) \mapsto \left\{ \begin{array}{l} \underset{\sim}{a}^i(\underline{\xi}, \underline{\eta}, \underline{\zeta}, \underset{\sim}{z}) \in \mathbb{R}^3 \\ \beta^i(\underline{\xi}, \underline{\eta}, \underline{\zeta}, \underset{\sim}{z}) \in \mathbb{R} \end{array} \right\}, \quad i = 1, \ldots, m,$$

(8.15) $$\mathbb{R}^m \times \partial\mathcal{B} \ni (\underline{\xi}, \underset{\sim}{z}) \mapsto \gamma^i(\underline{\xi}, \underset{\sim}{z}) \in \mathbb{R}, \quad i = 1, \ldots, m$$

satisfy

(8.16a) For almost all $\underset{\sim}{z}$ in $\mathcal{B}$, the functions $\underset{\sim}{a}^i(\cdot,\cdot,\cdot,\underset{\sim}{z})$, $\beta^i(\cdot,\cdot,\cdot,\underset{\sim}{z})$ are continuous and for all $\underline{\xi},\underline{\eta},\underline{\zeta}$, the functions $\underset{\sim}{a}^i(\underline{\xi},\underline{\eta},\underline{\zeta},\cdot)$, $\beta^i(\underline{\xi},\underline{\eta},\underline{\zeta},\cdot)$ are measurable. (These are the Carathéodory conditions.)

(8.16b) For almost all $\underset{\sim}{z} \in \partial\mathcal{B}$, the functions $\gamma^i(\cdot,\underset{\sim}{z})$ are continuous and for all $\underline{\xi}$, the functions $\gamma^i(\underline{\xi},\cdot)$ are measurable on $\partial\mathcal{B}$ (with respect to two-dimensional Lebesgue measure).

(8.16c) There exist a constant $c_1 > 0$ and a function $k_1 \in L_{p*}(\mathcal{B})$ (with $p^* = p/(p-1)$) such that

$$|\underset{\sim}{a}^i(\underline{\xi},\underline{\eta},\underline{\zeta},\underset{\sim}{z})|,\ |\beta^i(\underline{\xi},\underline{\eta},\underline{\zeta},\underset{\sim}{z})| \le c_1[|\xi|^{p-1} + |\eta|^{p-1} + |\zeta|^{p-1} + k_1(\underset{\sim}{z})]$$

for $i = 1,\ldots,m$; $\alpha = 1,2,3$.

The Hölder inequality then implies that the functions

$$\underset{\sim}{a}^i(\underline{u}(\cdot), \frac{\partial \underline{v}}{\partial \underset{\sim}{z}}(\cdot), \hat{\underset{\sim}{k}}(\underline{u}, \frac{\partial \underline{u}}{\partial \underset{\sim}{z}})(\cdot),\cdot),\ \beta^i(\underline{u}, \frac{\partial \underline{v}}{\partial \underset{\sim}{z}}(\cdot), \hat{\underset{\sim}{k}}(\underline{u}, \frac{\partial \underline{u}}{\partial \underset{\sim}{z}})(\cdot),\cdot)$$

are in $L_{p*}(\mathcal{B})$ for all $\underline{u},\underline{v} \in (W_p^1)^m$. It follows that the functional

$$a(\underline{u},\underline{w}) \equiv \int_{\mathcal{B}} [\sum_i \underset{\sim}{a}^i(\underline{u}(\underset{\sim}{z}), \frac{\partial \underline{u}}{\partial \underset{\sim}{z}}(\underset{\sim}{z}), \hat{\underset{\sim}{k}}(\underline{u}, \frac{\partial \underline{u}}{\partial \underset{\sim}{z}})(\underset{\sim}{z}),\underset{\sim}{z}) \cdot \frac{\partial w_i}{\partial \underset{\sim}{z}}(\underset{\sim}{z})$$
$$+ \sum_i \beta^i(\underline{u}(\underset{\sim}{z}), \frac{\partial \underline{u}}{\partial \underset{\sim}{z}}(\underset{\sim}{z}), \hat{k}(\underline{u}, \frac{\partial \underline{u}}{\partial \underset{\sim}{z}})(\underset{\sim}{z}),\underset{\sim}{z}) w_i(\underset{\sim}{z})] dv(\underset{\sim}{z})$$
$$+ \int_{\partial\mathcal{B}} \sum_i \gamma^i(\underline{u}(z)) w_i(z) da(z) \tag{8.17}$$

is well defined for all $\underline{u},\underline{w}, \in W_p^1(\mathcal{B})$.

We shall prescribe $u_1,\ldots,u_m$ respectively on subsets $\mathcal{S}_1,\ldots,\mathcal{S}_m$ of $\partial\mathcal{B}$. We assume that these subsets are measurable. Let V be the closed subspace of $[W_p^1(\mathcal{B})]^m$ containing $[\mathring{W}_p^1(\mathcal{B})]^m$ that consists of functions $(w_1,\ldots,w_m)$ for which $w_1 = 0$ on $\mathcal{S}_1,\ldots,w_m = 0$ on $\mathcal{S}_m$ in the sense of trace. Let $\bar{\underline{u}}$ be a

given element of $[W_p^1(\mathcal{B})]^m$. We require that u_1 agree with $\bar{u}_1$ on S_1, etc., in the sense of trace by seeking solutions $\underline{u}$ of our equations in $[W_p^1(\mathcal{B})]^m$ for which $\underline{u} - \bar{\underline{u}} \in V$. (This prescription of boundary conditions enables us to avoid the very delicate questions of whether functions defined on $S_1,\ldots,S_m$ can be extended to functions in $W_p^1(\mathcal{B})$.)

Since $V \ni \underline{w} \mapsto a(\underline{u},\underline{w})$ is a bounded linear functional for each $\underline{u}$ in $W_p^1(\mathcal{B})$, the Riesz Representation Theorem enables us to write

$$a(\underline{u},\underline{w}) = \langle \underline{A}(\underline{u}),\underline{w}\rangle \,, \tag{8.18}$$

where $\langle \underline{v},\underline{w}\rangle \equiv \int_{\mathcal{B}} \underline{v}\cdot\underline{w}\, dv$. If the $\underset{\sim}{a}^i$ are continuously differentiable and if $\underline{u}$ is twice continuously differentiable on $\mathcal{B}$ and vanishes on $\partial\mathcal{B}$, then $\underline{A}(\underline{u}) = A^1(\underline{u}),\ldots,A^m(\underline{u}))$ where

$$A^i(\underline{u}) \equiv -\mathrm{Div}\, \underset{\sim}{a}^i(\underline{u}, \frac{\partial \underline{u}}{\partial \underset{\sim}{z}}, \hat{\underline{k}}(\underline{u}, \frac{\partial \underline{u}}{\partial \underset{\sim}{z}}, \underset{\sim}{z})(\underset{\sim}{z}), \underset{\sim}{z}) + \beta^i(\underline{u}, \frac{\partial \underline{u}}{\partial \underset{\sim}{z}}, \underline{k}(\underline{u}, \frac{\partial \underline{u}}{\partial \underset{\sim}{z}}, \underset{\sim}{z})(\underset{\sim}{z}), \underset{\sim}{z}) \,. \tag{8.19}$$

Let us set $\underline{\eta} = (\underset{\sim}{\eta}_1,\ldots,\underset{\sim}{\eta}_m)$. Our basic abstract result is the following:

<u>8.20. Theorem.</u> <u>Let</u> $\partial\mathcal{B}$ <u>have a locally Lipschitz continuous graph.</u> <u>Let</u> $p \in (1,\infty)$. <u>Let</u> (8.16) <u>hold.</u> <u>Suppose that</u>

$$\frac{a(\underline{v},\underline{v})}{\|\underline{v},V\|} \to \infty \quad \underline{\text{as}} \quad \|\underline{v},V\| \to \infty \quad \underline{\text{for}} \quad \underline{v} \in V \,, \tag{8.21}$$

$$\sum_i [\underset{\sim}{a}^i(\underline{\xi},\underline{\eta} + \underline{\rho},\underline{\zeta},\underset{\sim}{z}) - \underset{\sim}{a}^i(\underline{\xi},\underline{\eta},\underline{\zeta},\underset{\sim}{z})]\cdot\underset{\sim}{\rho}_i > 0 \;\; \forall \underline{\rho} \equiv (\underset{\sim}{\rho}_1,\ldots,\underset{\sim}{\rho}_m) \neq \underline{0} \,. \tag{8.22}$$

<u>If</u> $\mathcal{B}$ <u>is bounded, let</u>

$$\sum_i \underset{\sim}{a}^i(\underline{\xi},\underline{\eta},\underline{\zeta},\underset{\sim}{z})\cdot\underset{\sim}{\eta}_i[|\underline{\eta}| + |\underline{\eta}|^{p-1}]^{-1} \to \infty \quad \text{as} \quad |\underline{\eta}| \to \infty \tag{8.23a}$$

<u>for almost all</u> $\underset{\sim}{z}$ <u>in</u> $\mathcal{B}$ <u>and for bounded</u> $\underline{\xi},\underline{\eta}$. <u>If</u> $\mathcal{B}$ <u>is unbounded, let the</u>

following stronger restriction hold: There is a number $c_2 > 0$ and a function $k_2 \in L_1(\mathcal{B})$ such that

(8.24b) $$\sum_i \underset{\sim}{a}^i(\underline{\xi},\underline{\eta},\underline{\zeta},\underset{\sim}{z})\cdot\underset{\sim}{\eta}_i > c_2|\underline{\eta}|^p - k_2(\underset{\sim}{z}) \ .$$

Define $\tilde{\underline{k}}$ by

(8.25) $$[W_p^1(B)]^m \ni \underline{u} \mapsto \tilde{\underline{k}}(\underline{u})(\cdot) \equiv \hat{k}(\underline{u},\partial\underline{u}/\partial\underset{\sim}{z})(\cdot)$$

where $\hat{\underline{k}}$ is defined in (10.13). Let χ_C be the characteristic function of a set C in $\mathcal{B}$. For every subdomain C of $\mathcal{B}$ with compact closure in $\mathcal{B}$ let

(8.26) $$[W_p^1(B)]^m \ni \underline{u} \mapsto \chi_C(\cdot)\tilde{k}(\underline{u})(\cdot) \in [L_p(C)]^r$$

be compact. Then for every $\underline{f} \in V^*$ and for every $\bar{\underline{u}} \in [W_p^1(\mathcal{B})]^m$ there exists a $\underline{u} \in [W_p^1(\mathcal{B})]^m$ with $\underline{u} - \bar{\underline{u}} \in V$ such that

(8.27) $$\langle\underline{A}(\underline{u}),\underline{v}\rangle = \langle\underline{f},\underline{v}\rangle \quad \forall\underline{v} \in V \ .$$

The proof of this theorem is obtained by making minor adjustments to those of Brezis (1968) (cf. Lions (1969, p. 297)) and Browder (1977). We note the following points: In a bounded domain the operator A is of the "calculus of variations type" because of its monotonicity in the local values of its highest order derivatives and because of its compactness (through $\tilde{\underset{\sim}{k}}$) in the global values of the highest order derivatives. Since our integral operator (7.6) for constant electric currents is not compact on unbounded domains, we had to use the theory of Browder (1977) based upon the compactness of (8.26) to support our intended applications.

We identify the variables appearing in Theorem 8.20 with those used in the problem outlined at the beginning of this subsection. In particular, we set

(8.28) $$\underline{u} = (\lambda,\varphi,\psi),\ \tilde{\underline{k}}(\underline{u})(\cdot) = \hat{k}\left(\frac{\partial\lambda}{\partial\underset{\sim}{z}},\ \lambda,\ \frac{\partial\varphi}{\partial\underset{\sim}{z}},\ \frac{\partial\psi}{\partial\underset{\sim}{z}}\right)(\cdot) \ ,$$

where $\hat{k}$ is defined in (5.6). We identify the variables appearing in (8.17) with those of (7.8):

(8.29) $$\underset{\sim}{a}^1(\underline{u}(z),\ \frac{\partial\underline{u}}{\partial\underset{\sim}{z}}(\underset{\sim}{z}),\tilde{\underline{k}}(\underline{u})(\underset{\sim}{z}),\underset{\sim}{z}) = \hat{\underset{\sim}{g}}\left(\frac{\partial\lambda}{\partial\underset{\sim}{z}}(\underset{\sim}{z}),\lambda(\underset{\sim}{z}),\frac{\partial\varphi}{\partial\underset{\sim}{z}}(\underset{\sim}{z}),\frac{\partial\psi}{\partial\underset{\sim}{z}}(\underset{\sim}{z}) + \hat{h}(\underline{\Delta})(\underset{\sim}{z}),\underset{\sim}{z}\right) \ ,$$

$$\underset{\sim}{a}^2(\underline{u}(z),\ \frac{\partial\underline{u}}{\partial\underset{\sim}{z}}(\underset{\sim}{z}),\tilde{\underline{k}}(\underline{u})(\underset{\sim}{z}),\underset{\sim}{z}) = \hat{\underset{\sim}{d}}\left(\frac{\partial\lambda}{\partial\underset{\sim}{z}}(\underset{\sim}{z}),\lambda(\underset{\sim}{z}),\frac{\partial\varphi}{\partial\underset{\sim}{z}}(\underset{\sim}{z}),\frac{\partial\psi}{\partial\underset{\sim}{z}}(\underset{\sim}{z}) + \hat{h}(\underline{\Delta})(\underset{\sim}{z}),\underset{\sim}{z}\right) \ ,$$

$$\underset{\sim}{a}^3(\underline{u}(z),\ \frac{\partial\underline{u}}{\partial\underset{\sim}{z}}(\underset{\sim}{z}),\tilde{\underline{k}}(\underline{u})(\underset{\sim}{z}),\underset{\sim}{z}) = \hat{\underset{\sim}{b}}\left(\frac{\partial\lambda}{\partial\underset{\sim}{z}}(\underset{\sim}{z}),\lambda(\underset{\sim}{z}),\frac{\partial\varphi}{\partial\underset{\sim}{z}}(\underset{\sim}{z}),\frac{\partial\psi}{\partial\underset{\sim}{z}}(\underset{\sim}{z}) + \hat{h}(\underline{\Delta})(\underset{\sim}{z}),\underset{\sim}{z}\right) \ ,$$

$$\beta^1(\underline{u}(z), \frac{\partial \underline{u}}{\partial \underset{\sim}{z}}(\underset{\sim}{z}), \underline{\tilde{k}}(\underline{u})(\underset{\sim}{z}), \underset{\sim}{z}) = -\hat{\rho}\left(\frac{\partial \lambda}{\partial \underset{\sim}{z}}(\underset{\sim}{z}), \lambda(\underset{\sim}{z}), \frac{\partial \varphi}{\partial \underset{\sim}{z}}(\underset{\sim}{z}), \frac{\partial \psi}{\partial \underset{\sim}{z}}(\underset{\sim}{z}) + \hat{h}(\underline{\Delta})(\underset{\sim}{z}), \underset{\sim}{z}\right) ,$$

$$\beta^2(\underline{u}(z), \frac{\partial \underline{u}}{\partial \underset{\sim}{z}}(\underset{\sim}{z}), \underline{\tilde{k}}(\underline{u})(\underset{\sim}{z}), \underset{\sim}{z}) = \sigma(\underset{\sim}{z}) ,$$

$$\beta^3(\underline{u}(z), \frac{\partial \underline{u}}{\partial \underset{\sim}{z}}(\underset{\sim}{z}), \underline{\tilde{k}}(\underline{u})(\underset{\sim}{z}), \underset{\sim}{z}) = 0 ,$$

$$\gamma^1(\underline{u}, \underset{\sim}{z}) = -\bar{\gamma}(\lambda(\underset{\sim}{z}), \varphi(\underset{\sim}{z}), \psi(\underset{\sim}{z}), \underset{\sim}{z}) ,$$

$$\gamma^2(\underline{u}, \underset{\sim}{z}) = -\bar{\delta}(\lambda(\underset{\sim}{z}), \varphi(\underset{\sim}{z}), \psi(\underset{\sim}{z}), \underset{\sim}{z}) ,$$

$$\gamma^3(\underline{u}, \underset{\sim}{z}) = -\bar{\beta}(\lambda(\underset{\sim}{z}), \varphi(\underset{\sim}{z}), \psi(\underset{\sim}{z}), \underset{\sim}{z}) .$$

We identify $\underline{w}$ with $(\lambda^{\#}, \varphi^{\#}, \psi^{\#})$. Note that hypothesis (8.22) is ensured by the restricted strong ellipticity condition. We then have

8.30. Theorem. *Let* $\hat{\underset{\sim}{g}}, \hat{\underset{\sim}{d}}, \hat{\underset{\sim}{b}}, \hat{\rho}, \bar{\gamma}, \bar{\delta}, \bar{\beta}$ *satisfy the hypotheses of Theorem 8.20 with the identifications* (8.28) *and* (8.29). *Then* (7.8) *with* $\underset{\sim}{\chi}^{\#} = \underset{\sim}{0}$ *is satisfied for all* $(\lambda^{\#}, \varphi^{\#}, x^{\#})$ *in* V.

The question of regularity of solutions for the types of systems described in this section remains open. Giaquinta (1983) gives partial regularity results for more restricted systems. However, it is by no means clear how much regularity is physically reasonable for either of the more general types of problems presented here. Ball (1982) suggests that discontinuous solutions of problems such as those treated in (8.7) can be used to model rupture of solid bodies. In addition, we suggest above that operators such as (8.13) can be used in constitutive equations to model the nonlocal behavior of ferromagnetic materials, and the physical evidence of so-called "domain structures" (cf. Tebble (1969)) suggests that highly discontinuous magnetic fields are to be expected from a good model of such materials.

Part II. The Semi-Inverse Problem

9. Formulation of the Semi-Inverse Problem

Let $\{\underset{\sim}{i}_1, \underset{\sim}{i}_2, \underset{\sim}{i}_3\}$ be a fixed right-handed orthonormal basis for $\mathbb{E}^3$ and let $\underline{x} = (s,\theta,z)$ be the set of cylindrical polar coordinates for $\mathbb{E}^3$ defined by

$$(9.1) \qquad \underset{\sim}{z} = \tilde{\underset{\sim}{z}}(\underline{x}) \equiv s\underset{\sim}{k}_1(\theta) + z\underset{\sim}{k}_3(\theta)$$

where

$$(9.2) \qquad \underset{\sim}{k}_1(\theta) = \cos\theta\underset{\sim}{i}_1 + \sin\theta\underset{\sim}{i}_2,\ \underset{\sim}{k}_2(\theta) = -\sin\theta\underset{\sim}{i}_1 + \cos\theta\underset{\sim}{i}_2,\ \underset{\sim}{k}_3(\theta) = \underset{\sim}{i}_3 .$$

Let $\tilde{\underline{x}}$ denote the usual inverse of $\tilde{\underset{\sim}{z}}$ so that (9.1) is equivalent to $\underline{x} = \tilde{\underline{x}}(\underset{\sim}{z})$. Each triple $\underline{x}$ also identifies a material point. We set

$$(9.3) \qquad \tilde{\underset{\sim}{y}}(\underline{x}) \equiv \underset{\sim}{y}(\tilde{\underset{\sim}{z}}(\underline{x})), \quad \text{etc.}$$

We consider semi-inverse problems in which $\tilde{\underset{\sim}{y}}, \tilde{\lambda}, \tilde{\varphi}, \tilde{\underset{\sim}{h}}$ have the form

$$(9.4a) \qquad \tilde{\underset{\sim}{y}}(\underline{x}) = w_1(s)\underset{\sim}{e}_1(\underline{x}) + [w_3(s) + \alpha_{32}\theta + \alpha_{33}z]\underset{\sim}{e}_3(\underline{x})$$

with

$$(9.4b) \qquad \underset{\sim}{e}_1(\underline{x}) = \cos\omega(\underline{x})\underset{\sim}{i}_1 + \sin\omega(\underline{x})\underset{\sim}{i}_2,$$

$$\underset{\sim}{e}_2(\underline{x}) = -\sin\omega(\underline{x})\underset{\sim}{i}_1 + \cos\omega(\underline{x})\underset{\sim}{i}_2,\ \underset{\sim}{e}_3(\underline{x}) = \underset{\sim}{i}_3 ,$$

$$(9.4c) \qquad \omega(\underline{x}) = w_2(x) + \alpha_{22}\theta + \alpha_{23}z ,$$

$$(9.4d) \qquad \tilde{\lambda}(\underline{x}) = w_4(s) ,$$

$$(9.4e) \qquad \tilde{\varphi}(\underline{x}) = w_5(s) + \alpha_{52}\theta + \alpha_{53}z ,$$

$$(9.4f) \qquad \tilde{\underset{\sim}{h}}(\underline{x}) = h_i(s)\underset{\sim}{k}_i(\theta) .$$

(Here i is summed from 1 to 3.) We shall make constitutive assumptions on $\hat{\underset{\sim}{j}}$ to ensure that $\tilde{\Psi}(\underline{x})$ (cf. Sec. 5) has the form

$$(9.4g) \qquad \tilde{\Psi}(\underline{x}) = w_6(s) + \alpha_{62}\theta + \alpha_{63}z .$$

We take the body to be

$$(9.5) \qquad \mathcal{B} = \tilde{\underset{\sim}{z}}([a,1] \times [-\Theta,\Theta] \times [-Z,Z])$$

with $0 < a < 1,\ 0 < \Theta \leq \pi,\ Z > 0$. Then $\mathcal{B}$ is a cylindrical tube (possibly slit) if $\Theta = \pi$ and is a sector thereof if $\Theta < \pi$. For simplicity we do not treat the interesting and technically complicated case that $a = 0$; the methods

for doing so are virtually identical to those used by Antman (1983). The deformations defined by (9.4a-c) constitute "family 2" of Truesdell & Noll (1965, Sec. 59). The other functions of (9.4) are so specified as to ensure that our final problem consists of ordinary functional differential equations.

The chain rule implies that

$$\text{(9.6a)}\quad \underset{\sim}{F}(\tilde{\underset{\sim}{z}}(\underline{x})) = \frac{\partial \underset{\sim}{y}}{\partial \underset{\sim}{z}}(\tilde{\underset{\sim}{z}}(\underline{x})) = \frac{\partial \tilde{\underset{\sim}{y}}}{\partial \underline{x}}(\underline{x}) \cdot \frac{\partial \tilde{\underline{x}}}{\partial \underset{\sim}{z}}(\tilde{\underset{\sim}{z}}(\underline{x}))$$
$$= [w_1'(s)\underset{\sim}{e}_1(\underline{x}) + w_1(s)w_2'(s)\underset{\sim}{e}_2(\underline{x}) + w_3'(s)\underset{\sim}{e}_3]\underset{\sim}{k}_1(\theta)$$
$$+ s^{-1}[\alpha_{22}w_1(s)\underset{\sim}{e}_2(\underline{x}) + \alpha_{32}\underset{\sim}{e}_3]\underset{\sim}{k}_2(\theta) + [\alpha_{23}w_1(s)\underset{\sim}{e}_2(\underline{x}) + \alpha_{33}\underset{\sim}{e}_3]\underset{\sim}{k}_3 ,$$

$$\text{(9.6b)}\quad \underset{\sim}{g}(\tilde{\underset{\sim}{z}}(\underline{x})) = \frac{\partial \lambda}{\partial \underset{\sim}{z}}(\tilde{\underset{\sim}{z}}(\underline{x})) = w_4'(s)\underset{\sim}{k}_1(\theta) ,$$

$$\text{(9.6c)}\quad \underset{\sim}{e}(\tilde{\underset{\sim}{z}}(\underline{x})) = \frac{\partial \varphi}{\partial \underset{\sim}{z}}(\tilde{\underset{\sim}{z}}(\underline{x})) = w_5'(s)\underset{\sim}{k}_1(\theta) + \alpha_{52}s^{-1}\underset{\sim}{k}_2(\theta) + \alpha_{53}\underset{\sim}{k}_3 ,$$

$$\text{(9.6d)}\quad \frac{\partial \psi}{\partial \underset{\sim}{z}}(\tilde{\underset{\sim}{z}}(\underline{x})) = w_6'(s)k_1(\theta) + \alpha_{62}s^{-1}k_2(\theta) + \alpha_{63}\underset{\sim}{k}_3 .$$

The representation (9.6a) reduces (3.2) to the requirement that

$$\text{(9.7a)}\quad (\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32})(w_1/s)w_1' > 0 \quad \text{a.e.}$$

Since w_1 is a radial distance, we require that

$$\text{(9.7b)}\quad w_1(s) > 0 \quad \text{for} \quad s \in [a,1] ,$$

whence (9.7a) reduces to

$$\text{(9.7c)}\quad (\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32})w_1' > 0 \quad \text{a.e.}$$

For simplicity we require that

$$\text{(9.7d)}\quad w_1' > 0 \quad \text{a.e.;}$$

the opposite case corresponds to an eversion (cf. Antman (1979)) and provides no further technical difficulties.

Note that the components of (9.6), (9.4d,f) with respect to the indicated base vectors and dyads are independent of θ and z. (It is easy to show that (9.4a) and (9.4e) are the most general forms whose gradients have this property.) We denote the ordered set of the components of $\underline{\Gamma}$ corresponding to (9.4), (9.6) by the single symbol

$$(9.8)\qquad \underline{\gamma}(s) = (\gamma_1(s),\ldots,\gamma_{15}(s))$$
$$\equiv (w_1'(s), w_1(s)w_2'(s), w_3'(s), \alpha_{22}s^{-1}w_1(s), \alpha_{32}s^{-1}, \alpha_{23}w_1(s), \alpha_{33},$$
$$w_4'(s), w_4(s), w_5'(s), \alpha_{52}s^{-1}, \alpha_{53}, h_1(s), h_2(s), h_3(s)) .$$

We define the physical components of the dependent constitutive variables by

$$(9.9a,b)\qquad \hat{T}_{ij}(\underline{\gamma},s) \equiv \underset{\sim}{e}_i(\underline{x})\cdot\hat{\underset{\sim}{T}}(\underline{\Gamma},\underset{\sim}{z})\cdot\underset{\sim}{k}_j(\theta),\quad \hat{q}_j(\underline{\gamma},s) \equiv \hat{\underset{\sim}{q}}(\underline{\Gamma},\underset{\sim}{z})\cdot\underset{\sim}{k}_j(\theta) ,$$

$$(9.9c,d)\qquad \hat{d}_j(\underline{\gamma},s) \equiv \hat{\underset{\sim}{d}}(\underline{\Gamma},\underset{\sim}{z})\cdot\underset{\sim}{k}_j(\theta),\quad \hat{b}_j(\underline{\gamma},s) \equiv \hat{\underset{\sim}{b}}(\underline{\Gamma},\underset{\sim}{z})\cdot\underset{\sim}{k}_j(\underline{\gamma},s) ,$$

$$(9.9e)\qquad \hat{j}_j(\underline{\gamma},s) \equiv \hat{\underset{\sim}{j}}(\underline{\Gamma},\underset{\sim}{z})\cdot\underset{\sim}{k}_j(\theta)$$

when $\underline{\Gamma}$ has the form corresponding to (9.4) and (9.6), assuming that the constitutive functions $\hat{\underset{\sim}{T}}$, etc., are such that these constitutive functions for the physical components depend only on $\underline{\gamma}$ and s. These representations are valid when the constitutive functions $\hat{\underset{\sim}{T}}$, etc., are hemitropic and depend on $\underset{\sim}{z}$ only through s. They are also valid for special forms of aeolotropy. It then follows that $\hat{\underset{\sim}{j}}\cdot\partial\varphi/\partial\underset{\sim}{z}$ depends only on $\underline{\gamma}$ and s. We also assume that σ depends only on s.

We now obtain an alternative representation for $\underset{\sim}{h}$ in terms of $\partial\psi/\partial\underset{\sim}{z}$ directly for the semi-inverse problem; the specialization of the results of Section 5 does not yield the new representation. Substituting (9.4f) and (9.9e) into (3.10) we get

$$(9.10)\qquad 0 = \hat{j}_1(\underline{\gamma}(s),s) ,$$

$$(9.11)\qquad [sh_2(s)]' = s\hat{j}_3(\underline{\gamma}(s),s),\quad h_3'(s) = -\hat{j}_2(\underline{\gamma}(s),s) .$$

Thus $\underset{\sim}{h}$ must have the form

$$(9.12a,b)\qquad h_1(s) = w_6'(s),\quad h_2(s) = s^{-1}[\alpha_{62} - \int_s^1 t\hat{j}_3(\underline{\gamma}(t),t)dt] ,$$

$$(9.12c)\qquad h_3(s) = \alpha_{63} + \int_s^1 \hat{j}_2(\underline{\gamma}(t),t)dt .$$

Therefore $\underset{\sim}{h}$ can be written as the sum of the gradient of (9.4g) and an integral operator (cf. (5.2)).

Condition (9.10) may be regarded either as a restriction on $\underline{\gamma}$ or else as being identically satisfied by virtue of choosing the constitutive function $\hat{j}_1$ to be the zero function, in which case the material is incapable of conducting electricity in the radial direction. In the former case we assume that (9.10) can be uniquely solved for w_5' in terms of the other elements of $\underline{\gamma}$:

(9.13) $$w_5'(s) = \hat{e}_1(\underline{\gamma}^-(s),s) \ ,$$

where $\underline{\gamma}^-$ stands for all the components of $\underline{\gamma}$ except w_5'. A sufficient condition for (9.13) to be equivalent to (9.10) is that $w_5' \mapsto \hat{j}_1(\underline{\gamma},s)$ be strictly increasing and assume both negative and positive values. That this function be strictly increasing is ensured by the strict monotonicity of $\underset{\sim}{e} \mapsto \hat{\underset{\sim}{j}}(\underline{\Gamma},\underset{\sim}{z})$. Equations (9.10) and (9.13) are also equivalent in the important special case that $\hat{j}_1(\underline{\gamma},s)$ has the same sign as w_5', which occurs, e.g., if $\hat{\underset{\sim}{j}}(\underline{\Gamma},\underset{\sim}{z}) = J(\underline{\Gamma},\underset{\sim}{z})\underset{\sim}{e}$ where J is a positive-valued *scalar* function. In this case (9.13) reduces to $w_5' = 0$.

Let

(9.14) $$\underline{w} = (w_1,\ldots,w_6), \quad \underline{\alpha} = (\alpha_{22},\ldots,\alpha_{63}) \ .$$

By controlling the dependence of $\hat{j}_2$ on $\hat{j}_3$ and h_2 and h_3 we can imitate the development of Section 5 to show that (9.12b,c) can be uniquely solved for h_2 and h_3 in terms of the other variables. Thus we can replace these equations with

(9.15a) $$h_2(s) = s^{-1}[\alpha_{62} + \kappa_2(\underline{w}(\cdot),\underline{\alpha},s)] \ ,$$

(9.15b) $$h_3(s) = \alpha_{63} + \kappa_3(\underline{w}(\cdot),\underline{\alpha},s) \ .$$

Alternatively we may observe that (9.12b,c) is equivalent to an initial value problem for h_2,h_3. If we assume that there is a number $p > 1$ such that $\hat{j}_2(\underline{\gamma}(\cdot),\cdot)$ and $\hat{j}_3(\underline{\gamma}(\cdot),\cdot)$ are integrable on $[a,1]$ when $\underline{\gamma} \in L_p([a,1])$

and that there is a number $K(\underline{w},\underline{\alpha})$ such that

$$\sum_{\alpha,\beta=1} |\partial\hat{j}_\alpha(\underline{\gamma}(s),s)/\partial h_\beta| < K(\underline{w},\underline{\alpha})$$

when $\underline{\gamma} \in L_p([a,1])$, then the standard theory of ordinary differential equations (cf. Hale (1969, Secs. I.5, I.6)) implies that (9.12b,c) has a unique absolutely continuous solution on $[a,1]$, which we can represent by (9.15a,b). Note that this result does not require restrictions like those of Theorem 5.3 on the size of $\partial\hat{\mathbf{j}}/\partial\mathbf{h}$ and on the size of the domain. The Arzela-Ascoli Theorem implies that $W_p^1((a,1)) \ni \underline{w}(\cdot) \mapsto \kappa_2(\underline{w}(\cdot),\underline{\alpha},\cdot),\ \kappa_3(\underline{w}(\cdot),\underline{\alpha},\cdot) \in C^0([a,1])$ are compact (when this construction of h_2 and h_3 is used).

We henceforth assume that the representation (9.15) is valid and that κ_2 and κ_3 have this compactness property.

We are now ready to write down the governing equations for our semi-inverse problem when the only body force, the Lorentz force, is absorbed into the effective stress and when the only heat source is that of (3.6), due to Joule heating. Let $\ell_i \mathbf{e}_i$ be the axial vector corresponding to $\mathbf{L}$. Let us set

(9.16) $$\hat{\underline{\xi}} = (\hat{\xi}_1,\ldots,\hat{\xi}_6),\quad \hat{\underline{\eta}} = (\hat{\eta}_1,\hat{\eta}_2,0,\hat{\eta}_4,\hat{\eta}_5,0)$$

with

(9.17a) $$\hat{\underline{\xi}}(\underline{w}',\underline{w},\underline{\alpha},\underline{v}(\cdot),s) \equiv (\hat{T}_{11}, w_1\hat{T}_{21}, \hat{T}_{31}, \hat{q}_1, \hat{d}_1, \hat{b}_1)\ ,$$

(9.17b) $$\hat{\eta}_1(\underline{w}',\underline{w},\underline{\alpha},\underline{v}(\cdot),s) \equiv \hat{T}_{21}w_2' + \alpha_{22}s^{-1}\hat{T}_{22} + \alpha_{23}\hat{T}_{23}\ ,$$

(9.17c) $$\hat{\eta}_2(\underline{w}',\underline{w},\underline{\alpha},\underline{v}(\cdot),s) \equiv \hat{\ell}_3\ ,$$

(9.17d) $$\hat{\eta}_4(\underline{w}',\underline{w},\underline{\alpha},\underline{v}(\cdot),s) \equiv \alpha_{52}s^{-1}\hat{j}_2 + \alpha_{53}\hat{j}_3\ ,$$

(9.17e) $$\hat{\eta}_5(s) = -\sigma\ ,$$

where the arguments of the constitutive functions appearing on the right sides of (9.17a-d) are $\underline{\gamma}, s$ and with every h_1,h_2,h_3 appearing on these right sides replaced by w_6', $\alpha_{62}s^{-1} + \kappa_2(\underline{v}(\cdot),\underline{\alpha},s)$, $\alpha_{63} + \kappa_3(\underline{v}(\cdot),\underline{\alpha},s)$ respectively. Note the definition of $\hat{\xi}_2$. In line with the remark following (3.12) there is no loss

of physical content in taking $\eta_2 = \ell_3 = 0$. We do so because it simplifies the ensuing analysis. Then by using the componential form of (3.12), we reduce the governing equation (5.13)-(5.16) to the following system of ordinary-functional differential equations for $\underline{w}, \underline{\alpha}$:

$$(s\hat{\underline{\xi}})' = s\hat{\underline{\eta}} = \underline{0} \tag{9.18}$$

where the arguments of $\hat{\underline{\xi}}$ and $\hat{\underline{\eta}}$ are $\underline{w}', \underline{w}, \underline{\alpha}, \underline{w}(\cdot), s$. (We have introduced our constitutive functions in (9.17) with the argument $\underline{v}(\cdot)$ so as to avoid confusion in Section 10 when we take certain partial derivatives of these functions.)

If (9.10) is equivalent to (9.13), then w_5' is completely determined by the other components of $\underline{w}$ and $\underline{\alpha}$, which can be found from the remaining equations and side conditions. We accordingly discard the fifth equation of (9.18), which is

$$(s\hat{d}_1)' = s\sigma \ . \tag{9.19}$$

We regard this equation as determining the σ necessary to maintain the semi-inverse state (9.4). This interpretation of (9.19) smells fishy, but is in fact quite reasonable: Consider, e.g., constitutive equations of the form

$$j_1 = Je_1, \quad d_1 = De_1$$

where J and D are positive-valued scalar functions. Then (9.13) and (9.19) require that $\sigma = 0$. Thus when (9.10) is equivalent to (9.13), we shall simply ignore (9.19), regarding (9.18) as the suitably truncated system. We shall comment on boundary conditions below.

If $\hat{j}_1$ is the zero function, then we need take no action with respect to (9.19).

We now specify boundary conditions. Our prescription is compatible with the formalism of Section 7. On the cylindrical face $s = 1$ of $\partial\mathcal{B}$ we either fix the outer radius:

(9.20a) $$w_1(1) = \bar{w}_1(1)$$

where $\bar{w}_1(1)$ is a given positive number or else we prescribe the traction:

(9.20b) $$\hat{\xi}_1(\underline{w}'(1),\underline{w}(1),\underline{\alpha},\underline{w}(\cdot),1) = \bar{\xi}_1(1)$$

where $\bar{\xi}_1(1)$ is a given number. (More generally, we could replace $\bar{\xi}_1(1)$ with $\bar{\xi}_1(\underline{w}(1),\underline{\alpha})$ where the new $\bar{\xi}_1$ is a prescribed function. Since only minor technical difficulties are introduced by such a replacement in this and other such Neumann conditions, we do not bother to pursue such generality.) We fix the deformation to within a rigid displacement by setting

(9.21) $$w_2(1) = 0 ,$$

(9.22) $$w_3(1) = 0 .$$

On this face we either prescribe the temperature:

(9.23a) $$w_4(1) = \bar{w}_4(1)$$

where $\bar{w}_4(1)$ is a given number or we prescribe the heat flux:

(9.23b) $$\hat{\xi}_4(\underline{w}'(1),\underline{w}(1),\underline{\alpha},\underline{w}(\cdot),1) = \bar{\xi}_4(1)$$

where $\bar{\xi}_4(1)$ is a given number. Finally we fix the data of the potentials φ and ψ by taking

(9.24) $$w_5(1) = 0 ,$$

(9.25) $$w_6(1) = 0 .$$

On the cylindrical face $s = a$ we prescribe alternative boundary conditions expressed in an analogous notation:

(9.26a,b) $$w_i(a) = \bar{w}_i(a) \quad \text{or} \quad \hat{\xi}_i(\underline{w}'(a),\underline{w}(a),\underline{\alpha},\underline{w}(\cdot),a) = \bar{\xi}_i(a) \quad \text{for } i = 1,\dots,6$$

where $\bar{w}_i(a)$, $i = 1,\dots,6$ and $\bar{\xi}_i(a)$, $i = 1,3,4,5,6$ are given constants and where

(9.26c) $$\bar{\xi}_2(a) \equiv w_1(a)\tau$$

with τ a given constant. The form of $\bar{\xi}_2(a)$ reflects its definition and the fact that it is a torque. In conformity with the condition that $w_1' > 0$, we require that $\bar{w}_1(1) > \bar{w}_1(a)$ when both these numbers are prescribed. Note that

(9.13) implies that

$$w_5(1) - w_5(a) = \int_a^1 \hat{e}_1(\underline{\gamma}^-(s),s)ds \tag{9.27}$$

so we are not free to prescribe both $w_5(1)$ and $w_5(a)$ when (9.13) holds.

To avoid dealing with the minor technical difficulties that can arise when all boundary conditions are of the Neumann type, we assume that the temperature w_4 is prescribed on at least one of the faces $s = a$ and $s = 1$. We need make no such provision for the variable w_1 because the growth conditions we shall impose on our constitutive functions preclude any trouble with coercivity ultimately due to Neumann conditions. If $\mathcal{B}$ is an entire tube, then $\Theta = \pi$. If we require that $\underset{\sim}{y}$, φ, and ψ be continuous, then

$$\alpha_{22} = \pm 1,\ \alpha_{32} = 0,\ \alpha_{52} = 0,\ \alpha_{62} = 0\ . \tag{9.28}$$

We obtain various kinds of dislocations by suspending (9.28a,b). If $\mathcal{B}$ is a sector of a tube, i.e., if $\Theta < \pi$, or if $\mathcal{B}$ is a slit tube i.e., if $\Theta = \pi$ but with the faces $\theta = -\pi$ and $\theta = \pi$ not identified, then we can prescribe certain degenerate boundary conditions on the faces $\theta = \pm\Theta$. We likewise prescribe such conditions on $z = \pm Z$.

We adopt the following alternative conditions for the faces $\theta = \pm\Theta$:

(9.29a,b) $\quad \alpha_{22} = \bar{\alpha}_{22}$ or

$$\hat{A}_{22}[\underline{w},\underline{\alpha}] \equiv \int_a^1 w_1(s)\hat{T}_{22}(\underline{\gamma}(s),s)ds = \bar{A}_{22}[w_1] \equiv \int_a^1 w_1(s)\bar{T}_{22}(s)ds$$

where $\bar{\alpha}_{22}$ is a given number and $\bar{A}_{22}$ is a given functional of w_1 having the indicated form. In the argument $\underline{\gamma}(s)$ of $\hat{T}_{22}$, $h_1(s)$ is replaced by (9.12a) and $h_2(s)$, $h_3(s)$ by (9.15). It is easy to see that $-Z\hat{A}_{22}[\underline{w},\underline{\alpha}]$ is the resultant effective torque about $\underset{\sim}{e}_3$ on the material face $\theta = -\Theta$ needed to

maintain the state (9.4) (cf. Antman (1983)). Similarly we prescribe

$$\text{(9.30a,b)} \qquad \alpha_{32} = \bar{\alpha}_{32} \quad \text{or} \quad \hat{A}_{32}[\underline{w},\underline{\alpha}] \equiv \int_a^1 \hat{T}_{32}(\underline{\gamma}(s),s)ds = \bar{A}_{32} ,$$

$$\text{(9.31a,b)} \qquad \alpha_{52} = \bar{\alpha}_{52} \quad \text{or} \quad \hat{A}_{52}[\underline{w},\underline{\alpha}] \equiv \int_a^1 d_2(\underline{\gamma}(s),s)ds = \bar{A}_{52} ,$$

$$\text{(9.32a,b)} \qquad \alpha_{62} = \bar{\alpha}_{62} \quad \text{or} \quad \hat{A}_{62}[\underline{w},\underline{\alpha}] \equiv \int_a^1 b_2(\underline{\gamma}(s),s)ds = \bar{A}_{62} .$$

Here $\underline{\gamma}$ has the form just described. $-Z\hat{A}_{32}[\underline{w},\underline{\alpha}]$ is the effective resultant force in the $\underset{\sim}{e}_3$-direction on the material face $\theta = -\Theta$. $\bar{A}_{32},\bar{A}_{52},\bar{A}_{62}$ are just numbers.

For the faces $z = \pm Z$, we likewise prescribe

$$\text{(9.33a,b)} \qquad \alpha_{23} = \bar{\alpha}_{23} \quad \text{or} \quad \hat{A}_{23}[\underline{w},\underline{\alpha}] \equiv \int_a^1 sw_1(s)\hat{T}_{23}(\underline{\gamma}(s),s)ds = \bar{A}_{23}[w_1] ,$$

$$\text{(9.34a,b)} \qquad \alpha_{33} = \bar{\alpha}_{33} \quad \text{or} \quad \hat{A}_{33}[\underline{w},\underline{\alpha}] \equiv \int_a^1 s\hat{T}_{33}(\underline{\gamma}(s),s)ds = \bar{A}_{33} ,$$

$$\text{(9.35a,b)} \qquad \alpha_{53} = \bar{\alpha}_{53} \quad \text{or} \quad \hat{A}_{53}[\underline{w},\underline{\alpha}] \equiv \int_a^1 sd_3(\underline{\gamma}(s),s)ds = \bar{A}_{53} ,$$

$$\text{(9.36a,b)} \qquad \alpha_{63} = \bar{\alpha}_{63} \quad \text{or} \quad \hat{A}_{63}[\underline{w},\underline{\alpha}] \equiv \int_a^1 sb_3(\underline{\gamma}(s),s)ds = \bar{A}_{63} .$$

$-\Theta\hat{A}_{23}[\underline{w},\underline{\alpha}]$ is the resultant effective torque about $\underset{\sim}{e}_3$ and $-\Theta\hat{A}_{33}[\underline{w},\underline{\alpha}]$ is the resultant effective force in the $\underset{\sim}{e}_3$-direction on the face $z = -Z$.

10. Consequences of the Strong Ellipticity Condition

In this section $\hat{\underline{\xi}}$ and $\hat{\underline{\eta}}$ have the arguments listed in (9.17). Thus the derivative of $\hat{\eta}_1$ with respect to w_1, say, is a pure partial derivative; no differentiation with respect to $\underline{v}(\cdot)$ is required.

Since $\partial\hat{\underset{\sim}{T}}/\partial w_1' = (\partial\hat{\underset{\sim}{T}}/\partial\underset{\sim}{F}):\underset{\sim}{e}_1\underset{\sim}{k}_1$, etc., definition (6.1) and the strong ellipticity condition imply that

(10.1a) $$\underline{\nu}\cdot(\partial\hat{\underline{\xi}}/\partial\underline{w}')\cdot\underline{\nu} = \omega((\nu_1\underset{\sim}{e}_1 + \nu_2 w_1\underset{\sim}{e}_2 + \nu_3\underset{\sim}{e}_3)\underset{\sim}{k}_1, \nu_4\underset{\sim}{k}_1, \nu_5\underset{\sim}{k}_1, \nu_6\underset{\sim}{k}_1) > 0$$

for all $\underline{\nu} \equiv (\nu_1,\ldots,\nu_6) \neq \underline{0}$ when $w_1 > 0$. Slightly abusing the notation, we likewise obtain

(10.1b) $$\underline{\nu}\cdot\frac{\partial\hat{\underline{\xi}}}{\partial\underline{x}}\cdot\underline{\nu} > 0 \;\; \forall\underline{\nu} \neq \underline{0} \text{ when } w_1 > 0 \text{ where}$$

$$\underline{x} \equiv (w_1', w_1 w_2', w_3', w_4', w_5', w_6') .$$

Next we observe that

(10.2) $$w_1|\underset{\sim}{c}|^2 = \underset{\sim}{a}\cdot\underset{\sim}{F}\cdot\underset{\sim}{c},\ \hat{\eta}_1 = \underset{\sim}{a}\cdot\hat{\underset{\sim}{T}}\cdot\underset{\sim}{c}$$

with $\underset{\sim}{a} = \underset{\sim}{e}_2$, $\underset{\sim}{c} = w_2'\underset{\sim}{k}_1 + (\alpha_{22}/s)\underset{\sim}{k}_2 + \alpha_{23}\underset{\sim}{k}_3$, so that the strong ellipticity condition implies that

(10.3) $$\partial\hat{\eta}_1/\partial w_1 \equiv \underset{\sim}{a}\underset{\sim}{c}:(\partial\hat{\underset{\sim}{T}}/\partial w_1) \equiv (\underset{\sim}{a}\underset{\sim}{c}):(\partial\hat{\underset{\sim}{T}}/\partial\underset{\sim}{F}):\underset{\sim}{a}\underset{\sim}{c} > 0 .$$

(Note that $\underset{\sim}{a}$ and $\underset{\sim}{c}$ are not independent of $\underset{\sim}{a}\cdot\underset{\sim}{F}\cdot\underset{\sim}{c}/|\underset{\sim}{c}|$).)

Suppose that $\bar{\alpha}_{32}$ and $\bar{\alpha}_{33}$ are prescribed. Set

(10.4) $$\mu = \bar{\alpha}_{33}\alpha_{22} - \bar{\alpha}_{32}\alpha_{23},\ \nu = \bar{\alpha}_{32}\alpha_{22} + \bar{\alpha}_{33}\alpha_{23} ,$$

(10.5) $$\hat{M} = \bar{\alpha}_{33}\hat{T}_{22} - \bar{\alpha}_{32}s\hat{T}_{23},\ \hat{N} = \bar{\alpha}_{32}\hat{T}_{22} + \bar{\alpha}_{33}s\hat{T}_{23} .$$

We solve (10.4) for α_{22} and α_{23} in terms of μ and ν and substitute the resulting expressions into the arguments of $\hat{M}$ and $\hat{N}$. The strong ellipticity condition then implies that

(10.6) $$\begin{pmatrix} \partial\hat{M}/\partial\mu & \partial\hat{M}/\partial\nu \\ \partial\hat{M}/\partial\mu & \partial\hat{N}/\partial\nu \end{pmatrix} \text{ is positive-definite .}$$

The combination of (10.1b) and Hypothesis 6.6 supports a global implicit function theorem (based on degree theory) that ensures that the function

(10.7a) $$\underline{x} \mapsto \hat{\underline{\xi}}(\underline{w}',\underline{w},\underline{\alpha},\underline{v}(\cdot),s)$$

has a strictly monotone inverse

(10.7b) $$\underline{\xi} \mapsto \underline{f}(\underline{\xi},\underline{w},\underline{\alpha},\underline{v}(\cdot),s) \ .$$

In particular, f_1, which delivers w_1', is positive on its domain.

11. Growth Conditions and Function Spaces

We introduce some notation to be used in the rest of this paper. Let

(11.1) $$\hat{\zeta}_{22} \equiv s^{-1}w_1\hat{T}_{22},\ \hat{\zeta}_{23} \equiv w_1\hat{T}_{23},\ \hat{\zeta}_{32} = s^{-1}\hat{T}_{32},\ \hat{\zeta}_{33} = \hat{T}_{33},$$
$$\hat{\zeta}_{52} \equiv s^{-1}\hat{d}_2,\ \hat{\zeta}_{53} = \hat{d}_3,\ \hat{\zeta}_{62} \equiv s^{-1}\hat{b}_2,\ \hat{\zeta}_{63} = \hat{b}_3\ ,$$
$$\underline{\hat{\zeta}}\cdot\underline{\alpha_\#} \equiv \sum_{\mu,\nu} \hat{\zeta}_{\mu\nu}\alpha_{\#\mu\nu}\ ,$$

the summation being taken over $\mu = 2,3,5,6$, $\nu = 2,3$. Let

(11.2) $$\underline{\omega} \equiv (\underline{w},\underline{\alpha})\ .$$

We set

(11.3) $$\langle\underline{m}(\underline{\omega}),\underline{\omega_\#})\rangle \equiv \int_a^1 (\underline{\hat{\xi}}\cdot\underline{w}_\#' + \underline{\hat{\eta}}\cdot\underline{w_\#} + \underline{\hat{\zeta}}\cdot\underline{\alpha_\#})s\,ds$$
$$+ a\underline{\bar{\xi}}(a)\cdot\underline{w_\#}(a) - \underline{\bar{\xi}}(1)\cdot\underline{w_\#}(1) - \sum_{\mu\nu} \bar{A}_{\mu\nu}[w_1]\alpha_{\#\mu\nu}\ ,$$

where the arguments of $\underline{\hat{\xi}},\underline{\hat{\eta}},\underline{\hat{\zeta}}$ are $\underline{w}',\underline{w},\underline{\alpha},\underline{w}(\cdot),s$. Observe that

(11.4) $$\int_a^1 \underline{\hat{\zeta}}\cdot\underline{\alpha_\#}s\,ds = \sum_{\mu\nu} \hat{A}_{\mu\nu}[\underline{\omega}]\alpha_{\#\mu\nu}\ .$$

The weak form of the boundary value problem of Section 9 is

(11.5) $$\langle\underline{m}(\underline{\omega}),\underline{\omega_\#}\rangle = 0 \quad \text{"for all"} \quad \underline{\omega_\#}\ .$$

In the next section, we give precise interpretations to relatives of (11.5).

We pose the basic growth conditions in terms of a scalar function W, which might be interpreted as a sort of stored energy function. It allows us to replace standard L_p-spaces by related spaces better equipped to handle possible aeolotropy.

11.6. Hypothesis. There are numbers $q > p > 1$, $c_1 > 0$, $c_2 > 0$, $c_3 > 0$, $c_4 > 0$ and a function $\mathbb{R}^{15} \ni \underline{\gamma} \mapsto W(\underline{\gamma},s) \in \mathbb{R}$ having the following properties:

(11.7) $W(\cdot,s)$ <u>is strictly convex</u> ,

(11.8) $$W(\underline{0},s) = 0 \ ,$$

(11.9) $W(\cdot,s)$ <u>is invariant under the change of sign of any component of</u> $\underline{\gamma}$,

(11.10) $$c_1|\underline{\gamma}|^p - c_2 < W(\underline{\gamma},s) < c_3|\underline{\gamma}|^q + c_4 \ .$$

<u>Let</u> $\mathring{\underline{w}}$ <u>be an affine function satisfying whatever Dirichlet conditions from</u> (9.20)-(9.26) <u>are prescribed. Let</u> $\mathring{\underline{\alpha}}$ <u>be a vector of the form</u> $\underline{\alpha}$ <u>of</u> (9.14) <u>with its entries taken to be</u> $\bar{\alpha}_{22},\ldots$ <u>whenever these numbers are prescribed in</u> (9.29)-(9.36) <u>and otherwise taken to be arbitrary with</u> $\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32} > 0$. <u>Let</u> $\mathring{\underline{\gamma}}$ <u>be generated from</u> $\mathring{\underline{w}}$ <u>and</u> $\mathring{\underline{\alpha}}$ <u>by</u> (9.8), (9.12), (9.15). <u>Then</u>

(11.11) $$\hat{\underline{\xi}}\cdot(\underline{w}' - \mathring{\underline{w}}') + \hat{\underline{n}}\cdot(\underline{w} - \mathring{\underline{w}}) + \hat{\underline{\zeta}}\cdot(\underline{\alpha} - \mathring{\underline{\alpha}}) > W(\underline{\gamma},s) - W(\mathring{\underline{\gamma}},s)$$

<u>when the arguments of the constitutive functions on the left side of</u> (11.11) <u>are</u> $\underline{w}',\underline{w},\underline{\alpha},\underline{w}(\cdot),s$ <u>and with</u> $\underline{\gamma}$ <u>and</u> $\mathring{\underline{\gamma}}$ <u>expressed in terms of these variables by</u> (9.8), (9.12a), (9.15).

Remarks. Condition (11.9) is a sort of isotropy condition. Its provenance is described by Antman (1983, eq. (7.7)). The mechanical terms from (11.11) correspond to a certain stress power. This issue is likewise treated at great length by Antman (1983).

We now introduce function spaces naturally associated with W. Let n be a positive integer. We set

(11.12) $$G \equiv \{\underline{\gamma} : \int_a^1 sW(\underline{\gamma}(s),s)ds < \infty\} \ ,$$

(11.13) $$E \equiv \{(\underline{w},\underline{\alpha}) : \underline{\gamma} \in G\} \ ,$$

(11.14) $$E_n = \{(\underline{w},\underline{\alpha}) \in E : nw_1' > 1 \ \text{ a.e.}, \ \ nw_1(s) > s, \ n(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) > 1\} \ ,$$

(11.15) $$A \equiv \{(\underline{w},\underline{\alpha}) \in E, \ w_1' > 0, \ \ w_1 > 0 \ \text{ a.e.}, \ \ \alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32} > 0,$$
$$\text{a fixed subset of } \{\underline{w}(a),\underline{w}(1),\underline{\alpha}\} \text{ is prescribed as in Sec. 9}\} \ ,$$

(11.16) $$A_n \equiv A \cap E_n \ .$$

A is the set of <u>admissible functions</u>. Conditions (11.7)-(11.10) ensure that $W(\cdot,s)$ satisfies the Δ_2-condition of Orlicz space theory, whence it follows

that G is a reflexive separable Banach space satisfying

(11.17) $$L_q((a,1)) \subset G \subset L_p((a,1))$$

(cf. Krasnosel'skii & Rutitskii (1958)). Since some components of $\underline{\gamma}$ are products of components of $(\underline{w},\underline{\alpha})$, neither E nor E_n is a Banach space. It is easy, however, to construct a suitable Banach space for $(\underline{w},\underline{\alpha})$: Let $(\underline{\mathring{w}},\underline{\mathring{\alpha}}) \in E_n$, let $\underline{\gamma}^{\#}$ be defined by (9.8) with $(\mathring{w}_2,\ldots,\mathring{w}_6,\underline{\mathring{\alpha}})$ replacing $(w_2,\ldots,w_6,\underline{\alpha})$, and let $\underline{\gamma}$ be defined by (9.8) with $\mathring{w}_1$ replacing w_1. Define

(11.18) $$Y_n \equiv \{(\underline{w},\underline{\alpha}): \int_a^1 sW(\underline{\gamma}^{\#}(s),s)ds < \infty,\ \int_a^1 sW(\underline{\gamma}\ (s),s)ds < \infty\} .$$

<u>11.19. Proposition</u>. Y_n <u>is a reflexive separable Banach space</u>. E_n <u>and</u> A_n <u>are closed subsets of</u> Y_n. A_n <u>is not empty if</u> n <u>is sufficiently large</u>.)

The proof of this result is identical to that of Antman (1983, Prop. 7.25.)

We now refine (11.11):

<u>11.20. Hypothesis</u>. <u>There are positive constants</u> $c_5,c_6,c_7,c_8,\varepsilon$, <u>depending on</u> $\mathring{w}_1,\mathring{w}_2$, <u>such that</u>

(11.21) $$(w_1' - \mathring{w}_1')\hat{\xi}_1 + (w_1 - \mathring{w}_1)\hat{\eta}_1 > c_5(|w_1'|^p + |w_1/s|^p) - c_6(1 + |\underline{\gamma}|^{p-\varepsilon}) ,$$

<u>where the arguments of</u> $\hat{\xi}_1,\hat{\eta}_1$ <u>are those listed in</u> (9.17).

The preceding hypotheses ensure that the material is not too weak; the following hypothesis ensures that it is not too strong.

<u>11.22. Hypothesis</u>. <u>Let the constitutive function introduced in</u> (9.9) <u>depend only on</u> $\underline{\gamma}$,s. <u>Let</u> $\underset{\sim}{a}(\underline{x})$ <u>and</u> $\tilde{\underset{\sim}{a}}(\underline{x})$ <u>be vectors that are linear combinations of</u> $\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3$ <u>with coefficients depending only on</u> s <u>and let</u> $\underset{\sim}{c}(\underline{\gamma})$ <u>and</u> $\tilde{\underset{\sim}{c}}(\underline{x})$ <u>be vectors that are linear combinations of</u> $\underset{\sim}{k}_1,\underset{\sim}{k}_2,\underset{\sim}{k}_3$ <u>with coefficients depending only on</u> s. <u>Let</u> $\underset{\sim}{a}\cdot\underset{\sim}{F}\cdot\underset{\sim}{c} \mapsto \tilde{\underset{\sim}{a}}\cdot\hat{\underset{\sim}{T}}\cdot\tilde{\underset{\sim}{c}}$, $\underset{\sim}{g}\cdot\underset{\sim}{a} \mapsto \hat{\underset{\sim}{g}}\cdot\tilde{\underset{\sim}{a}}$, $\underset{\sim}{e}\cdot\underset{\sim}{a} \mapsto \hat{\underset{\sim}{d}}\cdot\tilde{\underset{\sim}{a}}$, $\underset{\sim}{h}\cdot\underset{\sim}{a} \mapsto \hat{\underset{\sim}{b}}\cdot\tilde{\underset{\sim}{a}}$ <u>be strictly increasing and let</u> $\ell^{\pm}(\underset{\sim}{ac})$ <u>be constant when</u> $\underset{\sim}{F}$ <u>has the form</u> (9.6). <u>Then there are continuous functions</u> $(\underline{\Gamma},\underset{\sim}{z}) \mapsto \underset{\sim}{T}^{+}(\underline{\Gamma},\underset{\sim}{z})$, $\underset{\sim}{g}^{+}(\underline{\Gamma},\underset{\sim}{z})$, $\underset{\sim}{d}^{+}(\underline{\Gamma},\underset{\sim}{z})$, $\underset{\sim}{b}^{+}(\underline{\Gamma},\underset{\sim}{z})$, $\rho^{+}(\underline{\Gamma},\underset{\sim}{z})$

with $\tilde{\underset{\sim}{a}}(\underset{\sim}{x})\cdot\underset{\sim}{T}^{+}(\underline{\Gamma}(\underset{\sim}{z}),\underset{\sim}{z})\cdot\tilde{\underset{\sim}{c}}(\underset{\sim}{x})$, $\underset{\sim}{g}^{+}(\underline{\Gamma}(\underset{\sim}{z}),z)\cdot\tilde{\underset{\sim}{a}}(\underset{\sim}{x})$, etc., depending only on s when $\underline{\Gamma}$ has the form corresponding to (9.4) and (9.6) such that

$$(11.23)\qquad \tilde{\underset{\sim}{a}}\cdot\underset{\sim}{T}^{+}(\underline{\Gamma},\underset{\sim}{z})\cdot\tilde{\underset{\sim}{c}} > \begin{cases} \delta(\underset{\sim}{ac})\tilde{\underset{\sim}{a}}\cdot\hat{\underset{\sim}{T}}(\underline{\Gamma},\underset{\sim}{z})\cdot\tilde{\underset{\sim}{c}} & \text{if } \partial\mathcal{D}(\underset{\sim}{ac}) \neq \emptyset , \\ |\tilde{\underset{\sim}{a}}\cdot\hat{\underset{\sim}{T}}(\underline{\Gamma},\underset{\sim}{z})\cdot\tilde{\underset{\sim}{c}}| & \text{if } \partial\mathcal{D}(\underset{\sim}{ac}) = \emptyset , \end{cases}$$

$$\underset{\sim}{g}^{+}\cdot\tilde{\underset{\sim}{a}} > |\hat{\underset{\sim}{g}}\cdot\tilde{\underset{\sim}{a}}|,\ \underset{\sim}{d}^{+}\cdot\tilde{\underset{\sim}{a}} > |\hat{\underset{\sim}{d}}\cdot\tilde{\underset{\sim}{a}}|,\ \underset{\sim}{b}^{+}\cdot\tilde{\underset{\sim}{a}} > |\hat{\underset{\sim}{b}}\cdot\tilde{\underset{\sim}{a}}| ,$$

$$\rho^{+} > |\hat{\underset{\sim}{j}}\cdot\underset{\sim}{e}|$$

when $\underline{\Gamma}$ has the form corresponding to (9.4) and (9.6). Let $\underline{\xi}^{+},\underline{\eta}^{+},\underline{\zeta}^{+}$ be expressed in terms of $\underset{\sim}{T}^{+},\ldots$ just as $\hat{\underline{\xi}},\hat{\underline{\eta}},\hat{\zeta}$ are expressed in terms of $\underset{\sim}{T},\ldots$ by (9.17) and (11.1). If $\underline{\omega},\underline{\omega}_{\#} \in E$, then $s \mapsto \underline{\xi}^{+}\cdot\underline{w}_{\#}'(s) + \underline{\eta}^{+}\cdot\underline{w}_{\#}(s) + \underline{\zeta}^{+}\cdot\underline{\alpha}_{\#}(s)$, where the arguments of $\underline{\xi}^{+},\underline{\eta}^{+},\underline{\zeta}^{+}$ are $\underline{w}'(s),\underline{w}(s),\underline{\alpha},\underline{w}(\cdot),s$, is integrable on $[a,1]$. Moreover, if $\underline{\omega}$ is confined to a subset of E corresponding to a bounded subset of G, then the corresponding $(\underline{\xi}^{+},\underline{\eta}^{+},\underline{\zeta}^{+})$ and $\underline{m}(\underline{\omega})$ generate elements confined to a bounded subset of G^{*}. In particular,

$$(11.24)\qquad \hat{\xi}_1 < \xi_1^{+},\ \hat{\eta}_1 < \eta_1^{+},\ \hat{M} < M^{+} ,$$

$$|\hat{\xi}_2| < \xi_2^{+},\ |\hat{\xi}_3| < \xi_3^{+},\ |\hat{N}| < N^{+}$$

where $s \mapsto \xi_1^{+}(\underline{w}'(s),\underline{w}(s),\underline{\alpha},\underline{w}(\cdot),s),\ldots$ are in $L_1([a,1])$ and are confined to a bounded subset of L_1 when their arguments correspond to $\underline{\gamma}$'s in a bounded subset of G.

Condition (11.24) restricts the response of $\hat{\eta}_1$ (as well as other functions) in tension. We now formulate an hypothesis to control its behavior in compression. It furnishes a quantitative statement of how $\hat{\eta}_1$ is influenced more by changes in w_1 than by changes in w_1'. Let the function with values $\eta_1^{\dagger}(s^{-1}w_1,w_4,\underline{\xi},\underline{\alpha},\underline{v}(\cdot),s)$ be the composite function obtained from $\hat{\eta}_1$ by using (10.7b) to replace its first set of arguments $\underline{w}'$ with those of $\underline{f}$ in (10.7b). Let $f_1^{\dagger}$ be f_1 with its arguments in the same order as those of $\eta_1^{\dagger}$.

11.25. Hypothesis. Let there be a number $C > 0$, an octuple $\underline{\alpha}$, a scalar-valued function w_4, a function $\underline{\xi}$ with values in R^6, and a function $\underline{v}$

with $(\underline{v},\underline{\alpha}) \in A$ *such that*

$$(11.26) \qquad |\underline{\alpha}| < C,\ \alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32} > 1/C,\ |w_4| < C,\ |\underline{\xi}| < C\ ,$$

$$\hat{\xi}_1(\underline{v}',\underline{v},\underline{\alpha},\underline{v}(\cdot),s) < C,\ |\hat{\xi}_j(\underline{v}',\underline{v},\underline{\alpha},\underline{v}(\cdot),s)| < C \quad \text{for} \quad j = 2,\dots,6\ .$$

Then there is a number $m > 0$ *(depending on* C*)* *such that*

$$(11.27a) \qquad (0,m) \ni u \mapsto f_1^{\dagger}(u,w_4,\underline{\xi},\underline{\alpha},\underline{v}(\cdot),s) \quad \text{is decreasing}\ ,$$

$$(11.27b) \qquad (0,m) \ni u \mapsto \eta_1^{\dagger}(u,w_4,\underline{\xi},\underline{\alpha},\underline{v}(\cdot),s) \quad \text{is increasing.}$$

Moreover

$$(11.28) \qquad \limsup_{\varepsilon \to 0} \int_a^1 \eta_1^{\dagger}(\varepsilon + s^{-1} \int_a^s f_1^{\dagger}(\varepsilon,w_4(t),\underline{\xi}(t),\underline{v}(\cdot),t)dt,$$

$$w_4(s),\underline{\xi}(s),\underline{v}(\cdot),s)ds = -\infty$$

for each fixed $x \in (a,1]$.

The motivation for this hypothesis is given by Antman (1983).

Our final growth condition is

11.29. Hypothesis. There is a positive constant c_9 *such that*

$$(11.30) \qquad |\bar{A}_{22}[w_1]| + |\bar{A}_{23}[w_1]| < c_9\|w_1,L_p\|\ .$$

12. Existence of Classical Solutions

In this section we prove that when the Strong Ellipticity Condition and growth conditions hold, then a certain set of the boundary value problems posed in Section 9 have regular solutions. We restrict the data prescribed in the alternatives (9.29), (9.30), (9.33), (9.34) to be one of the following nine sets

$$(12.1)\qquad \begin{gathered}(\alpha_{22},\alpha_{23},\alpha_{32},\alpha_{33})\ ,\\ (\hat{A}_{22},\alpha_{23},\alpha_{32},\alpha_{33}),\ (\alpha_{22},\hat{A}_{23},\alpha_{32},\alpha_{33}),\ (\alpha_{22},\alpha_{23},\hat{A}_{32},\alpha_{33})\ ,\\ (\alpha_{22},\alpha_{23},\alpha_{32},\hat{A}_{33}),\ (\hat{A}_{22},\hat{A}_{23},\alpha_{32},\alpha_{33}),\ (\hat{A}_{23},\alpha_{23},\hat{A}_{32},\alpha_{33})\ ,\\ (\alpha_{22},\hat{A}_{23},\alpha_{32},\hat{A}_{33}),\ (\alpha_{22},\alpha_{23},\hat{A}_{32},\hat{A}_{33})\end{gathered}$$

because the unprescribed variables from $(\alpha_{22},\alpha_{23},\alpha_{32},\alpha_{33})$ are then confined by (9.7) to an open half-line or open half-plane. It then follows that the corresponding $\mathcal{A}_n$ is a closed convex subset of $\mathcal{Y}_n$.

Since our present work generalizes that of Antman (1983), we emphasize only those aspects that are novel. His work may be consulted for motivations and further discussion of such matters as the data omitted in (12.1). Our presentation also tacitly corrects some flaws in his arguments.

Our basic result is

12.2. Theorem. *Let the monotonicity conditions* (10.1), (10.3), (10.6) *hold. Let the Growth Hypotheses* 6.6, 6.10, 11.6, 11.20, 11.22, 11.25, 11.29 *hold. Let one of the sets of data of* (12.1) *be prescribed. Then the corresponding boundary value problems of Section 9 have classical solutions.*

Proof.

Step I. Existence of a solution to a truncated variational inequality. We can write

$$(12.3)\qquad \langle \underline{m}(\underline{\omega}),\underline{\omega}_{\#}\rangle \equiv \langle \underline{n}(\underline{\omega},\underline{\omega}),\underline{\omega}_{\#}\rangle\ ,$$

$$(12.4)\qquad \langle \underline{n}(\underline{\omega}^1,\underline{\omega}^2),\underline{\omega}_{\#}\rangle \equiv \int_a^1 \{\hat{\underline{\xi}}((\underline{w}^1)'(s),\underline{w}^2(s),\underline{\alpha}^2,\underline{w}^2(\cdot),s)\cdot \underline{w}_{\#}'(s)$$

$$+ \hat{\underline{n}}((\underline{w}^2)'(s),\underline{w}^2(s),\underline{\alpha}^2,\underline{w}^2(\cdot),s)\cdot \underline{w}_{\#}(s)\}ds$$

$$+ a\bar{\underline{\xi}}(a)\cdot \underline{w}_{\#}(a) - \bar{\underline{\xi}}(1)\cdot \underline{w}_{\#}(1)$$

$$+ \Sigma\{\hat{A}_{\mu\nu}[\underline{\omega}^2] - \bar{A}_{\mu\nu}[w_1^2]\}\alpha_{\#\mu\nu}$$

where the last term is summed over $\mu = 2,3,5,6$, $\nu = 2,3$.

Since A_n is a closed convex subset of Y_n, since (10.1) ensures that $\underline{m}$ is semi-monotone on A_n in the sense that

$$(12.5)\qquad \langle \underline{n}(\underline{\omega}^1,\underline{\omega}^2) - \underline{n}(\underline{\omega}^2,\underline{\omega}^2),\underline{\omega}^1 - \underline{\omega}^2\rangle \geq 0 \quad \forall \underline{\omega}^1,\underline{\omega}^2 \in A_n ,$$

and since $\underline{\omega} \to \kappa_2(\underline{\omega},\cdot),\kappa_3(\underline{\omega},\cdot)$ are compact by assumption, we can use Hypothesis 11.22 to show that $\underline{m}$ is an operator of the "type of the Calculus of Variations" (cf. Lions (1969)) from A_n to the dual space Y_n^* of Y_n. Thus $\underline{m}$ is pseudo-monotone on A_n. Hypothesis 11.6 ensures that $\underline{m}$ is coercive. A theorem of Brezis (1968) (cf. Lions (1969, p. 297)) then implies that for n sufficiently large there exists an $\underline{\omega}_n \in A_n$ satisfying the variational inequality

$$(12.6)\qquad 0 \geq \langle \underline{m}(\underline{\omega}_n),\underline{\omega}_n - \tilde{\underline{\omega}}\rangle \quad \forall \tilde{\underline{\omega}} \in A_n .$$

<u>Step II. Bounds on</u> $\underline{\gamma}_n$.

Let $\tilde{\underline{\omega}} = \mathring{\underline{\omega}} = (\mathring{\underline{w}},\mathring{\underline{\alpha}})$ where $(\mathring{\underline{w}},\mathring{\underline{\alpha}})$ is defined in Hypothesis 11.6. Let $\mathring{\underline{\gamma}}$ and $\underline{\gamma}_n$ correspond to $\mathring{\underline{\omega}}$ and $\underline{\omega}_n$. Then (12.6) and (11.11) imply that

$$(12.7)\qquad 0 \geq \int_a^1 \{\underline{\xi}_n\cdot(\underline{w}_n' - \mathring{\underline{w}}') + \underline{n}_n\cdot(\underline{w}_n - \mathring{\underline{w}})\}ds$$

$$+ a\bar{\underline{\xi}}(a)\cdot(\underline{w}_n(a) - \mathring{\underline{w}}(a)]$$

$$- \bar{\underline{\xi}}(1)\cdot[\underline{w}_n(1) - \mathring{\underline{w}}(a)]$$

$$+ \sum_{\mu,\nu} \{\hat{A}_{\mu\nu}[\underline{\omega}_n] - \bar{A}_{\mu\nu}[w_{n1}]\}(\alpha_{n\mu\nu} - \mathring{\alpha}_{\mu\nu})$$

$$\geq \int_a^1 W(\underline{\gamma}_n(s),s)ds - \int_a^1 W(\mathring{\underline{\gamma}}(s),s)ds - \Lambda$$

where $\underline{\xi}_n(s) \equiv \hat{\underline{\xi}}(\underline{w}_n'(s),\underline{w}_n(s),\underline{\alpha}_n,\underline{w}_n(\cdot),s)$, etc., and

$$(12.8) \qquad \Lambda \equiv -\int_a^1 \sigma(s)[w_{n5}(s) - \mathring{w}_5(s)]s\,ds$$

$$+ \bar{\underline{\xi}}(1)\cdot[\underline{w}_n(1) - \underline{\mathring{w}}(1)] - a\bar{\underline{\xi}}(a)\cdot[\underline{w}_n(a) - \underline{\hat{w}}(a)]$$

$$+ \sum_{\mu,\nu} \bar{A}_{\mu\nu}[w_{n1}](\alpha_{n\mu\nu} - \mathring{\alpha}_{\mu\nu}) .$$

In the following development we let C represent a positive constant independent of n, which can always be estimated in terms of the available data. The meaning of C can vary with each appearance. Note that (9.21) implies that $w_{n2}(1) - \mathring{w}_2(1) = 0$. Conditions (9.26c) and (9.21), the positivity of w'_{n1}, and the Hölder inequality then imply that

$$(12.9) \qquad |a\bar{\underline{\xi}}_2(a)[w_{n2}(a) - \mathring{w}_2(a)]|$$

$$< Caw_{n1}(a)[|w_{n2}(a) - w_{n2}(1)| + 1]$$

$$< C\int_a^1 w_{n1}(s)|w'_{n2}(s)|s\,ds + Cw_{n1}(1)$$

$$< C\|w_{n1}w'_{n2},L_p\| + Cw_{n1}(1) .$$

In this way, by using (9.21)-(9.26), the Hölder and Poincaré inequalities, and the estimates (11.30) we find that

$$(12.10) \qquad \Lambda < C\{1 + \|\underline{\gamma}_n,L_p\| + \|w_{n1},W_p^1\|\} .$$

(For certain sets of data in (12.1), $\|w_{n1},W_p^1\| < C\|\underline{\gamma}_n,L_p\|$.) Combining (11.11) with (12.7), (12.9) we obtain

$$(12.11) \qquad \|\underline{\gamma}_n,L_p\|^p < C\{1 + \|\underline{\gamma}_n,L_p\| + \|w_{n1},W_p^1\|\} .$$

Next we take all the components of $\underline{\tilde{\omega}}$ except $\tilde{w}_1$ to equal the components of ω_n and we take $\tilde{w}_1 = \mathring{w}_1$. Then (12.6) and (11.22) likewise yield

$$(12.12) \qquad \|w_{n1},W_p^1\|^p < C\{1 + \|\underline{\gamma}_n,L_p\|^{p-\epsilon} + \|w_{n1},W_p^1\|\} .$$

Inequalities (12.11) and (12.12) imply that

$$(12.13a,b) \qquad \|\underline{\gamma}_n,L_p\| < C, \quad \|w_{n1},W_p^1\| < C ,$$

whence $\Lambda < C$. It follows from (12.7) that

$$\int_a^1 W(\underline{\gamma}_n(s),s)sds < C \ . \tag{12.14}$$

Thus, by the definition of the norm of G (by duality according to the theory of Orlicz spaces), we obtain

$$\|\underline{\gamma}_n, G\| < C \ . \tag{12.15}$$

We accordingly get corresponding bounds on all the components of $\underline{\omega}_n$ except w_{n2}, α_{n22}, α_{n23}. To bound these variables we need a uniform positive lower bound for w_{n1}.

Step III. Integral inequalities. We now make judicious choices for $\underline{\tilde{\omega}}$ in (12.6) in order to extract useful consequences from it.

If $\bar{\xi}_1(1)$ is prescribed in (9.20b), then we let $a < x < 1$, $0 < \varepsilon < x - a$, and set

$$\tilde{w}_1(s) \equiv \begin{cases} w_{n1}(s) & \text{for } a < s < x - \varepsilon \ , \\ w_{n1}(s) + [s - (x - \varepsilon)]/\varepsilon & \text{for } x - \varepsilon < s < x \ , \\ w_{n1}(s) + 1 & \text{for } x < s < 1 \ , \end{cases} \tag{12.16}$$

$$\underline{\tilde{w}} = (\tilde{w}_1(s), w_{n2}, \ldots, w_{n6}), \ \underline{\tilde{\alpha}} = \underline{\alpha}_n \ . \tag{12.17}$$

Then $\underline{\tilde{w}} = (\underline{\tilde{w}}, \underline{\tilde{\alpha}}) \in A_n$, when $\underline{\tilde{w}}$ and $\underline{\tilde{\alpha}}$ are given by (12.16) and (12.17). Substituting (12.16), (12.17) into (12.6) and letting $\varepsilon \downarrow 0$ we obtain

$$x\xi_{n1}(x) > \bar{\xi}_1(1) - \int_x^1 s\eta_{n1}(s)ds \tag{12.18}$$

for almost all x in $(a,1)$. Since the right-hand side of (12.18) is a continuous function of x, we can assume that (12.18) holds for all x in $(a,1)$.

If on the other hand $w_1(1)$ is prescribed to equal $\tilde{w}_1(1)$ by (9.20a), then we require a more delicate construction. Let $\mathcal{P}_n$ be the set of all $y \in (a,1]$ for which

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \int_{y-\varepsilon}^{y} w_{n1}'(s)ds \tag{12.19}$$

exists and exceeds $1/n$; let $\mathcal{P}_n^c$ be its complement in $[a,1]$. (The theory of differentiation ensures that (12.19) exists a.e. On $\mathcal{P}_n^c$, $w_{n1}'(s) = 1/n$ a.e. Below we show that the Lebesgue measure of $\mathcal{P}_n^c$ approaches 0 as $n \to \infty$. For now, all we require is that $\mathcal{P}_n$ not be empty.)

Let us choose $y \in \mathcal{P}_n$, $x \in (a,y)$ and

$$0 < \lambda < w_{n1}(y) - w_{n1}(x) - \frac{1}{n}(y - x) , \tag{12.20}$$

the rightmost term of which is positive by the definition of $\mathcal{P}_n$. Since

$$s \mapsto w_{n1}(s) + (y - s)/n \equiv \varphi_n(s) \tag{12.21}$$

is continuous, since $\varphi_n(y) = w_{n1}(y) > w_{n1}(y) - \lambda$, and since $\varphi_n(x) < w_{n1}(y) - \lambda$ by (12.20), the intermediate value theorem ensures that the equation

$$w_{n1}(t) + \lambda = w_{n1}(y) - (y - t)/n \tag{12.22}$$

has a solution $t_n(\lambda) \in [x,y)$. Since φ_n is nowhere decreasing on $[x,y]$, all solutions of (12.22) lie in an interval, which is closed because φ_n is continuous. Since $t_n(\lambda)$ satisfies (12.22) and since $y \in \mathcal{P}_n$, it follows that there is a positive number θ, depending on w_{n1} and y, such that

$$[y - t_n(\lambda)][n^{-1} + \theta] < \int_{\rho_n(\lambda)}^{\zeta} w_{n1}'(s)ds = [y - t_n(\lambda)]/n + \lambda , \tag{12.23}$$

which implies that

$$t_n(\lambda) \to y \quad \text{as} \quad \lambda \downarrow 0 \quad \text{(for fixed } n) . \tag{12.24}$$

When $w_1(1)$ is prescribed to equal $\bar{w}_1(1)$, we let $y \in \mathcal{P}_n$, $x \in (a,y)$, $\varepsilon \in (0,x-a)$ and take

$$\text{(12.25)} \qquad \tilde{w}_1(s) \equiv \begin{cases} w_{n1}(s) & \text{for } a < s < x-\varepsilon\ , \\ w_{n1}(s) + [s-(x-\varepsilon)]\lambda/\varepsilon & \text{for } x-\varepsilon < s < x\ , \\ w_{n1}(s) + \lambda & \text{for } x < s < t_n(\lambda)\ , \\ w_{n1}(y) - (y-s)/n & \text{for } t_n(\lambda) < s < y\ . \end{cases}$$

We define $\underline{\tilde{\omega}}$ by (12.25), (12.17), observing that $\underline{\tilde{\omega}} \in \mathcal{A}_n$. We substitute this $\underline{\tilde{\omega}}$ into (12.6), let $\varepsilon \downarrow 0$, and then let $\lambda \downarrow 0$ to obtain

$$\text{(12.26)} \qquad x\xi_{n1}(x) \geq y\xi_{n1}(y) - \int_x^y s\eta_{n1}(s)ds$$

for all y in $\mathcal{P}_n$ and for all $x \in (a,y)$.

By the simpler, classical version of the process leading to (12.18) or (12.26) we likewise obtain

$$\text{(12.27)} \qquad x\xi_{nj}(x) = \xi_{nj}(1) - \int_x^1 s\eta_{nj}(s)ds, \quad j = 2,\ldots,6\ ,$$

for almost all x in $(a,1)$.

To be specific in the rest of our analysis, we suppose that $\bar{\alpha}_{32}$ and $\bar{\alpha}_{33}$ are prescribed. (Thus we can exploit (10.4)-(10.6).) By substituting $\underline{\tilde{\omega}} = (\underline{w}_n, \underline{\tilde{\alpha}})$ with

$$\text{(12.28a)} \qquad \underline{\tilde{\alpha}} = (\alpha_{n22} + \bar{\alpha}_{33},\ \alpha_{n23} - \bar{\alpha}_{32}, \bar{\alpha}_{32}, \bar{\alpha}_{33})$$

and with

$$\text{(12.28b)} \qquad \underline{\tilde{\alpha}} = (\alpha_{n22} + \lambda\bar{\alpha}_{32},\ \alpha_{n23} + \lambda\bar{\alpha}_{33}, \bar{\alpha}_{32}, \bar{\alpha}_{33}),\ \lambda \in \mathbb{R}$$

into (12.6) we obtain from the arbitrariness of λ that

$$\text{(12.29a)} \qquad \begin{aligned} &\bar{\alpha}_{33}\hat{A}_{22}[\underline{\omega}_n] - \bar{\alpha}_{32}\hat{A}_{23}[\underline{\omega}_n] \geq \bar{\alpha}_{33}\bar{A}_{22}[w_{n1}] - \bar{\alpha}_{32}\bar{A}_{23}[w_{n1}]\ , \\ &\bar{\alpha}_{32}\hat{A}_{22}[\underline{\omega}_n] + \bar{\alpha}_{33}\hat{A}_{23}[\underline{\omega}_n] = \bar{\alpha}_{32}\bar{A}_{22}[w_{n1}] + \bar{\alpha}_{33}\bar{A}_{23}[w_{n1}]\ . \end{aligned}$$

Step IV. Uniform lower bounds for w'_{n1}, w_{n1}, μ_n. Inequality (12.15) enables us to use the second part of Hypothesis 11.22 supported by (10.2), (10.3) to deduce from (12.18) that

$$x\xi_{n1}(x) > -C \tag{12.30}$$

for almost all x in $(a,1]$, and from (12.26b) that

$$x\xi_{n1}(x) > y\xi_{n1}(y) - C \tag{12.31}$$

for almost all y in P_n and for almost all x in (a,y). Since (12.30) and (12.31) yield essential lower bounds, we regard these inequalities as holding for all such x and y. Since $\eta_2 = 0 = \eta_3$, and since (12.27) consequently implies that

$$(1 - a)\xi_{nj}(1) = \int_a^1 s\xi_{nj}(s)ds \quad \text{for} \quad j = 2,3 \ , \tag{12.32}$$

we obtain from (12.15), (12.27), and Hypothesis 11.22 that

$$|s\xi_{n2}(s)|, \ |s\xi_{n3}(s)| < C \quad \forall s \in [a,1] \ . \tag{12.33}$$

Inequalities (11.26), (12.13b) enable us to deduce from (12.29) that there is a $C > 0$ such that

$$\bar{\alpha}_{33}\hat{A}_{22}[\underline{\omega}_n] - \bar{\alpha}_{32}\hat{A}_{23}[\underline{\omega}_n] \equiv \int_a^1 w_{n1}(s)\hat{M}(\underline{\gamma}_n(s),s)ds > -C \ , \tag{12.34a}$$

$$|\bar{\alpha}_{32}\hat{A}_{22}[\underline{\omega}_n] + \bar{\alpha}_{33}A_{23}[\underline{\omega}_n]| \equiv \left|\int_a^1 w_{n1}(s)\hat{N}(\underline{\gamma}_n(s),s)ds\right| < C \ . \tag{12.34b}$$

The analysis in the rest of this step relies critically on Hypothesis 6.10. We identify the basis $\{\underline{E}_\tau\}$ thus:

$$(12.35)\qquad \underset{\sim}{E}_1 = \underset{\sim}{e}_1\underset{\sim}{k}_1,\ \underset{\sim}{E}_2 = \underset{\sim}{e}_1\underset{\sim}{k}_2,\ \underset{\sim}{E}_3 = \underset{\sim}{e}_1\underset{\sim}{k}_3,\ \underset{\sim}{E}_4 = \underset{\sim}{e}_2\underset{\sim}{k}_1\ ,$$

$$\underset{\sim}{E}_5 = \frac{\underset{\sim}{e}_2(s\bar{\alpha}_{33}\underset{\sim}{k}_2 - \bar{\alpha}_{32}\underset{\sim}{k}_3)}{(s^2\bar{\alpha}_{33}^{\,2} + \bar{\alpha}_{32}^{\,2})^{1/2}}\ ,\quad \underset{\sim}{E}_6 = \frac{\underset{\sim}{e}_2(s\bar{\alpha}_{32}\underset{\sim}{k}_2 + \bar{\alpha}_{33}\underset{\sim}{k}_3)}{(s^2\bar{\alpha}_{32}^{\,2} + \bar{\alpha}_{33}^{\,2})^{1/2}}$$

$$\underset{\sim}{E}_7 = \underset{\sim}{e}_3\underset{\sim}{k}_1,\ \underset{\sim}{E}_8 = \underset{\sim}{e}_3\underset{\sim}{k}_2,\ \underset{\sim}{E}_9 = \underset{\sim}{e}_3\underset{\sim}{k}_3\ .$$

The dual basis is given by

$$(12.36)\qquad \underset{\sim}{E}_5^* = \frac{(s^2\bar{\alpha}_{33}^{\,2} + \bar{\alpha}_{32}^{\,2})^{1/2}}{s(\bar{\alpha}_{32}^{\,2} + \bar{\alpha}_{33}^{\,2})}\underset{\sim}{e}_2(\bar{\alpha}_{33}\underset{\sim}{k}_2 - s\bar{\alpha}_{32}\underset{\sim}{k}_3)\ ,$$

$$\underset{\sim}{E}_6^* = \frac{(s^2\bar{\alpha}_{32}^{\,2} + \bar{\alpha}_{33}^{\,2})^{1/2}}{s(\bar{\alpha}_{32}^{\,2} + \bar{\alpha}_{33}^{\,2})}\underset{\sim}{e}_2(\bar{\alpha}_{32}\underset{\sim}{k}_2 + s\bar{\alpha}_{33}\underset{\sim}{k}_3)\ ,$$

$$\underset{\sim}{E}_\tau^* = \underset{\sim}{E}_\tau \quad \text{for} \quad \tau \neq 5,6\ .$$

Thus

$$(12.37)\qquad \underset{\sim}{F}:\underset{\sim}{E}_1 = w_1',\ \underset{\sim}{F}:\underset{\sim}{E}_2 = 0,\ \underset{\sim}{F}:\underset{\sim}{E}_3 = 0,\ \underset{\sim}{F}:\underset{\sim}{E}_4 = w_1w_2'\ ,$$

$$\underset{\sim}{F}:\underset{\sim}{E}_5 = w_1\mu(s^2\bar{\alpha}_{33}^{\,2} + \bar{\alpha}_{32}^{\,2})^{-1/2},\ \underset{\sim}{F}:\underset{\sim}{E}_6 = w_1\nu(s^2\bar{\alpha}_{32}^{\,2} + \bar{\alpha}_{33}^{\,2})^{-1/2}\ ,$$

$$\underset{\sim}{F}:\underset{\sim}{E}_7 = w_3',\ \underset{\sim}{F}:\underset{\sim}{E}_8 = s^{-1}\bar{\alpha}_{32},\ \underset{\sim}{F}:\underset{\sim}{E}_9 = \bar{\alpha}_{33}\ ,$$

$$(12.38)\qquad \hat{\underset{\sim}{T}}:\underset{\sim}{E}_1^* = \hat{T}_{11} = \xi_1,\ \hat{\underset{\sim}{T}}:\underset{\sim}{E}_2^* = \hat{T}_{12},\ \hat{\underset{\sim}{T}}:\underset{\sim}{E}_3^* = \hat{T}_{13},\ \hat{\underset{\sim}{T}}:\underset{\sim}{E}_4^* = \hat{T}_{21}\ ,$$

$$\hat{\underset{\sim}{T}}:\underset{\sim}{E}_5^* = \frac{(s^2\bar{\alpha}_{33}^{\,2} + \bar{\alpha}_{32}^{\,2})^{1/2}}{s(\bar{\alpha}_{32}^{\,2} + \bar{\alpha}_{33}^{\,2})}\hat{M},\ \hat{\underset{\sim}{T}}:\underset{\sim}{E}_6^* = \frac{(s^2\bar{\alpha}_{32}^{\,2} + \bar{\alpha}_{33}^{\,2})}{s(\bar{\alpha}_{32}^{\,2} + \bar{\alpha}_{33}^{\,2})}\hat{N}\ ,$$

$$\hat{\underset{\sim}{T}}:\underset{\sim}{E}_7^* = \hat{T}_{31},\ \hat{\underset{\sim}{T}}:\underset{\sim}{E}_8^* = \hat{T}_{32},\ \hat{\underset{\sim}{T}}:\underset{\sim}{E}_9^* = \hat{T}_{33}\ .$$

(We could alternatively take $\underset{\sim}{E}_4 = \underset{\sim}{e}_2\underset{\sim}{c}/|\underset{\sim}{c}|$ where $\underset{\sim}{c}$ is given by (10.2).)

We now obtain a lower bound w_1^* for w_{n1} that is positive on $(a,1]$. Suppose for the sake of contradiction that w_{n1} have no such lower bound. Then there would be an x in $(a,1]$ and a subsequence $\underline{\omega}_n$ such that $w_{n1}(x) \to 0$. Then $w_{n1} \to 0$ uniformly on $[a,x]$. The representation of $w_{n1}(x) - w_{n1}(a)$ as an integral of w_{n1}' over $[a,x]$ shows that $w_{n1}' \to 0$ in $L_1(a,x)$, whence $\underline{\omega}_n$

has a further subsequence with $w'_{n1} \to 0$ pointwise a.e. on $[a,x]$. It follows from (12.15) that $\{1,5\} \subset b$, $\{2,3,4,6,7,8,9\} \subset e$ for almost all $y \in [a,x]$. Condition (12.34a) ensures that no alternative of (6.12b) is tenable for almost all $y \in [a,x]$. Hence there is a function w_1^* such that

(12.39) $$w_{n1}(s) \geq w_1^*(s) > 0 \quad \forall s \in (a,1] \quad \text{and} \quad \forall n .$$

Note that we can define w_1^* by

(12.40) $$w_1^*(s) \equiv \inf w_{n1}(s) .$$

It follows that w_1^* is nowhere decreasing, for if $x < y$, then

(12.41) $$\begin{aligned} w_1^*(y) - w_1^*(x) &= \inf w_{n1}(y) - \inf w_{n1}(x) \\ &= \inf w_{n1}(y) + \sup(-w_{n1}(x)) \\ &\geq \inf(w_{n1}(y) - w_{n1}(x)) \geq 0 \end{aligned}$$

since w_{n1} is nowhere decreasing. If $\bar{w}_1(a)$ is prescribed, then $w_1^*(a) = \bar{w}_1(a) > 0$. Otherwise, we have yet to show that $w_1^*(a) > 0$.

A simple version of the preceding arguments shows that there is a $C > 0$ such that

(12.42) $$\mu_n \geq 1/C .$$

We now confront the weakness of (12.26) and (12.31) inhering in the membership of y in P_n, which conceivably could be sparsely distributed over $[a,1]$.

<u>12.43. Lemma</u>. <u>Let</u> $w_{n1}(1) = \bar{w}_1(1)$. <u>The Lebesgue measure of</u> P_n^c <u>approaches</u> 0 <u>as</u> $n \to \infty$.

Proof. Were the conclusion false, there would be a $C > 0$ and a subsequence $\{\underline{\omega}_n\}$ such that the measure of P_n^c exceeds C^{-1}. Condition (12.15) implies that for any $\varepsilon \in (0, \frac{1}{2}(1 - a))$ there is a subset Q_ε of $[a + \varepsilon, 1]$ of measure $1 - a - 2\varepsilon$ and a positive number C_ε such that the absolute value of each component of $\underline{y}_n(s)$ except $w'_{n1}(s)$ is bounded by C_ε when $s \in Q_\varepsilon$. Now we choose ε so small that the measure of $P_n^c \cap Q_\varepsilon \equiv R_n^c$ has a positive lower bound

(independent of n). We fix ε. Since $w_{n1} < \bar{w}(1)$, inequality (11.24) implies that

(12.44) $$\int_{R_n} w_{n1} M_n ds < C ,$$

so that (12.34a) yields

(12.45) $$\int_{R_n^c} w_{n1} M_n ds > -C .$$

The properties of Q_ε and the alternatives of (6.12b) imply that there is a sequence m_n of numbers with $m_n \to \infty$ as $n \to \infty$ such that $M_n < -m_n$ on R_n^c. But this inequality is incompatible with (12.45) if the measure of R_n^c has a positive lower bound independent of n. □

We next obtain a lower bound ξ_1^* for ξ_{n1} that is continuous on $[a,1)$. This bound is given by (12.30) when $\bar{\xi}_1(1)$ is prescribed. We prove this by showing that for any $z \in [a,1)$ there is a positive real number $h(z)$ such that

(12.46) $$\xi_{n1}(s) > -h(z) \quad \forall s \in [a,z] .$$

By choosing a sequence of such z's approaching 1 we obtain a sequence of constant lower bounds for ξ_n whose graphs are horizontal line segments in the (s,ξ_{n1})-plane. By joining parts of these segments with straight lines we readily construct a lower bound continuous on the half-open interval $[a,1)$. Suppose that for given z there were no such $h(z)$. Then there would be a sequence $x_n \in [a,z]$ such that $\xi_{n1}(x_n) \to -\infty$ as $n \to \infty$. Then (12.31) would imply that

(12.47) $$\xi_{n1}(y_n) \to -\infty \quad \forall y_n \in P_n \cap (z,1] ,$$

so that $1 \in \mathcal{a}$ for $(\underline{w}_n(y_n),\underline{\alpha}_n)$. Since μ_n is bounded by (12.15), (12.39), (12.42), condition (6.12a) would imply that $M_n(y_n) \to -\infty$, which is impossible by an argument like that centered on (12.45). Thus there is a function ξ_1^* continuous on $[a,1)$ such that

(12.48) $$\xi_{n1}(s) > \xi_1^*(s) \quad \forall s \in [a,1], \forall n .$$

We now obtain a lower bound for $\{w_n'\}$. Let $\underline{f}$ be defined by (10.7). Then (12.48) implies that

$$(12.49)\qquad \begin{aligned} w_{n1}'(s) &= f_1(\underline{\xi}_n(s),\underline{w}_n(s),\underline{\alpha}_n,\underline{w}_n(\cdot),s) \\ &> f_1(\xi_1^*(s),\xi_{n2}(s),\ldots,\xi_{n6}(s),\underline{w}_n(s),\underline{\alpha}_n,\underline{w}_n(\cdot),s) \\ &> \min_{k<n} f_1(\xi_1^*(s),\xi_{k2}(s),\ldots) \equiv \psi_n(s) . \end{aligned}$$

ψ_n is continuous on $[a,1)$. The equation (12.32), the bounds (12.33), and their analogs for $\xi_{n4},\xi_{n5},\xi_{n6}$, the embedding theorem, the bounds on μ_n and ν_n, and inequality (12.39) all show that on any compact subset of $(a,1)$ the sequence $\{\psi_n\}$ is uniformly bounded, bounded below by a positive function, equicontinuous, and decreasing. The Arzela-Ascoli Theorem implies that the whole sequence ψ_n converges uniformly on any compact subset of $(a,1)$ to a continuous limit function ψ^*, which is positive on $(a,1)$. (If $\bar{\xi}_1(1)$ is prescribed, then ψ^* is positive on $(a,1]$; if $\tilde{w}_1(a)$ is prescribed, then ψ^* is positive on $[a,1)$.) We thus have

$$(12.50)\qquad w_{n1}(s) > \frac{a}{n} + \int_a^s \psi^*(t)dt .$$

<u>Step V. Classical solutions</u>. Let $0 < \varepsilon < \frac{1}{2}(1 - a)$. Let g be any piecewise continuously differentiable function with $g(s) = 0$ for $s \in [a,a+\varepsilon] \cup [1-\varepsilon,1]$ and with $|g'| < \frac{1}{2}\psi^*$. Set

$$(12.51)\qquad \tilde{w}_1 = w_{n1} + g,\quad \underline{\tilde{w}} = (\tilde{w}_1,w_{n1},\ldots) .$$

Then for n sufficiently large, $\underline{\tilde{\omega}} \in A_n$ (since ψ^* has a positive lower bound on $[a+\varepsilon,1-\varepsilon]$). We substitute (12.51) into (12.6) and use the arbitrariness of g to obtain in place of (12.18) and (12.26) the equality

$$(12.52)\qquad x\xi_{n1}(x) = y\xi_{n1}(y) - \int_x^y s\eta_{n1}(s)ds \quad \forall x,y \in [a+\varepsilon,1-\varepsilon] .$$

We use $\hat{\underline{f}}$ of (10.7) to convert (12.52), (12.27) into a form yielding an explicit representation for $\underline{w}_n'$. By the standard boot-strap argument it follows that $\underline{\omega}_n$ generates a twice continuously differentiable solution of (9.18) on $[a + \varepsilon, 1 - \varepsilon]$ when n is sufficiently large. Indeed, if k is a positive integer, then the representations for $\underline{w}_n'$ supported by the estimates of Steps II and IV show that $\underline{w}_n'$ is uniformly bounded and equicontinuous on $[a + k^{-1}, 1 - k^{-1}]$ and has a subsequence $\{\underline{w}_{n,k}'\}$ that converges uniformly on $[a + k^{-1}, 1 - k^{-1}]$ while $\underline{\alpha}_{n,k}$ converges in $\mathbb{R}^8$. We assume without loss of generality that $\{\underline{w}_{n,k+1}'\}$ is subsequence of $\{\underline{w}_{n,k}'\}$. It follows that the diagonal subsequence $(\underline{w}_{n,n}, \underline{\alpha}_{n,n})$ converges in $C^1([a + \varepsilon, 1 - \varepsilon]) \times \mathbb{R}^8$ to a limit $(\underline{w}, \underline{\alpha})$ for every $\varepsilon \in (0, \frac{1}{2}(1 - a))$. It is easily verified that $(\underline{w}, \underline{\alpha})$ satisfies the differential equations (9.18) on $(a, 1)$. Next we replace (12.28a) with

$$\tilde{\underline{\alpha}} = (\alpha_{n22} + \lambda\bar{\alpha}_{33}, \alpha_{n23} - \lambda\bar{\alpha}_{32}, \bar{\alpha}_{32}, \bar{\alpha}_{33}) , \tag{12.52}$$

which is admissible for small negative λ by virtue of (12.42). Thus in place of (12.29a) we obtain the corresponding equality. By letting $n \to \infty$ through the diagonal subsequence in this modification of (12.29a) and in (12.29b) we find that $(\underline{w}, \underline{\alpha})$ satisfies the obvious limit form, whence we obtain

$$\hat{A}_{22}[\underline{\omega}] = \bar{A}_{22}[w_1], \quad \hat{A}_{23}[\underline{\omega}] = \bar{A}_{23}[w_1] . \tag{12.53}$$

We verify that other side conditions are met by a similar process.

If $\bar{\xi}_1(1)$ is prescribed, we can carry out our construction of $\underline{w}_{n,k}'$ on intervals of the form $[a + k^{-1}, 1]$, thereby obtaining the equality corresponding to (12.18) for the limit $(\underline{w}, \underline{\alpha})$. Hence $\xi_1(\underline{w}'(s), \underline{w}(s), \underline{\alpha}, w(\cdot), s) \to \bar{\xi}_1(1)$ as $s \to 1$.

From now on, $\{\underline{\omega}_n\}$ is understood to stand for the diagonal subsequence $\{\underline{\omega}_{n,n}\}$ or a further subsequence thereof.

We can now show that $\xi_{n1}(1)$ is bounded below. Note that (10.2), (10.5) imply that

(12.54) $$\eta_{n1} = w_{n1}^{-1} w'_{n2}\xi_{n2} + s^{-1}(\bar{\alpha}_{32}^{2} + \bar{\alpha}_{33}^{2})^{-1}(\mu_n M_n + \nu_n N_n) .$$

Hypothesis 11.22 or estimate (12.34b) together with (12.39) imply that $\{|N_n|\}$ is bounded by a fixed integrable function of s on any compact subinterval of $(a,1]$. Hypothesis 11.22 and estimate (12.34b) imply the same for $|M_n|$. Hypothesis 11.22 alone implies the same for ξ_{n2}. Our estimates for μ_n, ν_n and our representation for w'_{n2} then show that for each $x \in (a,1]$ there is a number $h(x)$ such that

(12.55) $$\left|\int_x^1 \eta_{n1}(s)ds\right| < h(x) .$$

Combining this estimate with the limiting equality

(12.56) $$x\xi_1(x) = y\xi_1(y) - \int_x^y s\eta_1(s)ds \quad \text{for} \quad a < x < y < 1$$

corresponding to (12.26) we find that $\xi_1(1) > -\infty$, which implies that $\xi_{n1}(1)$ is bounded below.

To show that

(12.57) $$\xi_1(\underline{w}'(s), \underline{w}(s), \underline{\alpha}, \underline{w}(\cdot), s) \to \bar{\xi}_1(a) \quad \text{as} \quad s \to a$$

when $\bar{\xi}_1(a)$ is prescribed, we require a positive lower bound for $w_{n1}(a)$ so that we can choose an $\underline{\tilde{\omega}}$ with $\tilde{w}_1(a) < w_{n1}(a)$ for large enough n. Indeed, using (12.56) in the limiting form of (12.6) we obtain

(12.58) $$0 < [\xi_1(a) - \bar{\xi}_1(a)][w_1(a) - \tilde{w}_1(a)]$$

so that

(12.59) $$\xi_1(a) < \bar{\xi}_1(a) .$$

We can now use (12.56) to conclude that (12.55) holds with $x = a$. Thus

$$\xi_{n1}(s) < \bar{\xi}_1(a) + C < C . \tag{12.60}$$

We now prove the existence of a lower bound for $w_{n1}(a)$. Since f_1 is increasing in ξ_1, we obtain from (12.59) that

$$w'_{n1}(s) < f_1^{\dagger}(s^{-1}w_{n1}(s), w_{n4}(s), C, \xi_{n2}(s), \ldots, \xi_{n6}(s), \underline{w}_n(\cdot), s) . \tag{12.61}$$

Now suppose for the sake of contradiction that there is a subsequence for which $w_n(a) \to 0$ as $n \to \infty$. Estimate (12.39) ensures that

$$s^{-1}w_{n1}(s) > s^{-1}w_1^*(s) > w_1^*(s) > a^{-1}w_{n1}(a) \tag{12.62}$$

for this subsequence for n sufficiently large. Property (11.27a) enables us to deduce from (12.60), (12.61) that

$$w'_{n1}(s) < f_1^{\dagger}(a^{-1}w_{n1}(a), w_{n4}(s), \ldots) \tag{12.63}$$

for s sufficiently close to a and for n sufficiently large. Hence

$$w_{n1}(s) < w_{n1}(a) + \int_a^s f_1^{\dagger}(a^{-1}w_{n1}(a), w_{n4}(t), \ldots)dt . \tag{12.64}$$

Property (11.27b) then yields

$$\eta_{n1}(s) < \eta_1^{\dagger}(a^{-1}w_{n1}(a) + s^{-1}\int_a^s f_1^{\dagger}(a^{-1}w_{n1}(a), w_{n4}(t), \ldots)dt, w_{n4}(s), \ldots) . \tag{12.65}$$

But then (12.55) contradicts (11.28). It follows that $w_1(a) > 1/C$ and that we can consequently choose $\tilde{w}_1(a)$ to reverse the inequality in (12.58). Hence (12.57) holds. The demonstration that other Neumann conditions at a are satisfied is routine. These results complete the proof of Theorem 12.2.

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In this paper we study the steady-state behavior of solids that can sustain mechanical, electromagnetic, and thermal effects. We examine a class of boundary-value problems for a quasilinear system of functional differential equations that is derived from a very general model for such materials. We propose a physically reasonable constitutive theory which leaves this system amenable to modern methods of partial differential equations. The principal

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assumption is a modified version of the strong ellipticity condition. In Part I we prove existence results for the general system under some special physical assumptions (rigidity and hyperelasticity). Our formulation admits non-local interactions caused by the magnetic "self-field" generated by the deformed, conducting body. In Part II we show the existence and regularity of solutions of a system of functional ordinary differential equations arising from a semi-inverse problem in a more comprehensive physical situation.

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